Reliability of Interconnection Networks

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I would like to dedicate this thesis to my loving parents Professor Shiying Wang and Mrs. Zhifen Mu.

Declaration

I hereby certify that the work embodied in the thesis is my own work, conducted under normal supervision. The thesis contains no material which has been accepted, or is being examined, for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made. I give consent to the final version of my thesis being made available worldwide when deposited in the University's Digital Repository, subject to the provisions of the Copyright Act 1968 and any approved embargo.

Mujiangshan Wang

June 2019

Acknowledgement of Authorship

I hereby certify that the work embodied in this thesis contains published papers of which I am a joint author. I have included a written declaration below endorsed in writing by my supervisor, attesting to my contribution to the joint publications. By signing below I confirm that

I contributed the main proofs of lemmas, theorems and corollaries, the discussion of other proofs and the proofreading to the papers entitled as follows:

[1] Wang, M., Lin, Y., and Wang, S. (2016). The 2-good-neighbor diagnosability of Cayley graphs generated by transposition trees under the PMC model and MM* model. *Theoretical Computer Science*, 628:92–100.

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Yuqing Lin

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Abstract

Graph is a type of mathematical model to study the relationships among entities. The theory on graphs is called Graph Theory. It started in 1736 and has 283 years of history since the paper was written by Leonhard Euler on the Seven Bridges of Königsberg.

In computer science, the term "Interconnection Networks" has been used to refer to a set of interconnected elements. For example, a computer network where computers was connected by wires or Internet of Things (IoT) is connected via wireless connection. There are two types of network: static and dynamic.

Static networks are hard-wired and their configurations do not change. The structure, which is also called topology signifies that the nodes are arranged in specific shape and the shape is maintained throughout the networks. In this thesis, we focus on the static networks.

In graph theory, graphs are used to model the topology of network, whether it is networks of communication, data organization, computational devices, the flow of computation. For instance, the link structure of a local area network can be represented by an undirected graph, in which the vertices represent computers and edges represent connections between two computers. A similar approach can be applied to problems in social media, travel, biology, computer design, mapping the progression of neuro-degenerative diseases, and many other fields. Graph models could be directed, undirected and weighted, depending on the properties of the network we are studying. Fault-tolerance of networks is an important property. Fault-tolerance is the property that enables a system to continue operating properly in the event of the failure of some (one or more faults) of its components. Fault-tolerance is particularly sought after in high-availability or life-critical systems.

We are interested in the fault-tolerance of networks. Considering the corresponding graph model of the networks, connectivity of the graphs measures how resistant a graph can be against the nodes (link) removal. In graph theory, there is a set of fault-tolerance related parameters, such as restricted-connectivity, extra-connectivity etc., which gave refined information about how robust is a network.

Performance of the distributed system is significantly determined by the choice of the network topology. Desirable properties of an interconnection network include low degree, low diameter, symmetry, low congestion, high connectivity, and high fault-tolerance. For the past several decades, there has been active research on a class of graphs called Cayley graphs because this type of graph possesses many of the above properties. Many Cayley graphs based on permutation groups has proven to be suitable for designing interconnection networks, such as Star graph [1, 2, 47], Hypercubes [8], Pancake graphs [2, 79], Shuffel-Exchange Permutation Network [50], the Rotation-Exchange Network [110]. These graphs are symmetric, regular, and share the desirable properties described above.

In this thesis, we studied the connectivity and diagnosability of some popular network structures. For instance, Cayley graphs generated by transpositions, expanded k-ary n-cube and locally twisted cube.

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Chapter 1

Introduction

1.1 Network

A network is a collection of connected objects. In mathematics, graphs are used to model the underlying structure of networks. The area of mathematics concerning the study of graphs is called graph theory.

Graphs can represent all sorts of networks in the real world. For example, one could describe the Internet as a network where the vertices are computers or other devices and the edges are physical (or wireless) connection between the devices. The World Wide Web is a huge network where the pages are vertices and hyper-links are the edges. Other examples include social networks, networks of publications linked by citations, transportation networks, metabolic networks, and communication networks.

An *update on a graph* is an operation that inserts or deletes edges or vertices of the graph or changes attributes associated with edges or vertices, such as cost or color. By *dynamic graph* we refer to the graph that is subject to a sequence of updates while *static graph* denote a graph without such updates.

We can classify dynamic graph problems according to the types of updates allowed. In particular, a dynamic graph problem is said to be *fully dynamic* if the updates include unrestricted insertions and deletions of edges or vertices. A dynamic graph problem is said to be *partially dynamic* if only one type of update, either insertions or deletions, is allowed.

Research on dynamic graph typically answers queries such as, whether the graph is connected or which is the shortest path between any two vertices.

In this thesis, we only focus on static graph, i.e., no updates are allowed. We use the term *graph* if no ambiguity arises. When uses graphs to model networks, one quickly realizes that the simple network model with identical vertices and edges cannot describe important features of real networks. For example, the simple graph is undirected. However, in the World Wide Web, for example, the links between pages are directed. Unfortunately, just because linking from a page to Wikipedia's main page doesn't mean that Wikipedia will put a link from their main page back to this page. Because the edges are directed in this way, we need to use a directed graph to present the World Wide Web. In such a directed graph (or digraph, for short), we typically draw the edges as arrows to indicate the direction.

In some networks, not all vertices and edges are created equal. For example, in metabolic networks, vertices may indicate different enzymes which have a wide variety of behaviors, and edges may indicate vastly different types of interactions. To model such difference, one can introduce different types of vertices and edges in the network. In networks where the differences among vertices and edges can be captured by a single number that, for example, indicates the strength of the interaction, weighted graph is a good model.

In some contexts, one may work with graphs that have multiple edges between the same pair of vertices. One might also allow a vertex to have a self-connection, meaning an edge from the vertex itself to itself.

In the thesis, we will focus primarily on unweighted graphs with vertex and edge without labels. In this rest of the thesis, we use network and (simple) unweighted graphs interchangeably.

1.2 Reliability

The reliability of a network (graph) is the capability of the network (graph) to continue working when a number of vertices or edges have failed. The larger number of faulty elements (vertices) or connections (edges) that a network can tolerate, the better is the network's reliability.

Reliability of networks can be measured in many ways using parameters such as connectivity or edge connectivity. Connectivity is one of the fundamental concepts of graph theory. It asks for the minimum number of elements (vertices or edges) whose removal leads to the disconnection of the graph.

When removing elements (vertices or edges) to disconnect the graph, a special case is that such vertices are all adjacent to one vertex or edges are all incident to one vertex, which means that a single vertex has been isolated from the rest of the graph. In this case, the connectivity doesn't give clear indication of the reliability of the whole of the network, as the rest of the network might have high reliability or fault tolerance, could still function if we ignore the isolated vertex.

To distinct the special case from the rest of the cases, there are all kinds of refined measurements, for example, in 1983, F. Harary [39] introduced the concept of *conditional connectivity* by requiring some properties for disconnected components of $G-F$, where F is a vertex set whose removal leads to the disconnection of graph G. On the other hand in 1988, A.H. Esfahanian and S.L. Hakimi [29] gave generalizations of edge-connectivity by specifying certain conditions to be satisfied by the disconnected components. For example, there is at least one cycle in each connected part, which is well applied in Ring topology for constructing networks.

In my study, I am using some recently proposed, practical oriented measurements such as g-good connectivity and g-extra connectivity. These new parameters better measure the robustness of networks.

Concerning the network topological properties, Cayley graph is highly symmetric, has well defined hierarchical structure, highly connected and with great fault tolerance [40]. Cayley Graphs become an attractive underlying topology of computer networks. For examples see [74].

Another family of graphs is Hypercube-like networks. Owing to nice properties such as logarithmic number of links per vertex and logarithmic diameter, high symmetry and high recursive constructability, linear bisection width, and exists simple efficient routing and broadcasting algorithms, the n-dimensional Hypercube Q_n has been one of the most popular interconnection network topologies [66]. On the other hand, as was shown by Hillis [42], hypercube do not have the smallest possible diameter. To achieve smaller diameter with the same number of vertices and links, a variety of hypercube variants were proposed [17, 22, 24, 26, 30, 41, 77]. Among these variations, Möbius cube [24], crossed cube [26], twisted cube [41], and Mcube [77] have diameters of about half of the diameter of a hypercube of the same size. A common feature of these variants is that the labels of some neighbor vertices may differ in a large number of bits. As a result, a portion of good properties of hypercube is lost in these variants. For example, the design of efficient parallel algorithms on these variants turns out to be a difficult task. In this thesis, we study two families of Hypercube-like networks, expanded k-ary n-cube and Locally Twisted Cube.

In order to keep as many nice properties of hypercube as possible, a better hypercube variant should be conceptually closer to hypercube than existing variants. Motivated by this intuition, we introduce a new hypercube variant. We call our topology as the n-dimensional locally twisted cube LTQ_n because its vertices can be one-to-one labeled with 0–1 binary sequences of length n, so that the labels of any two adjacent vertices differ in at most two successive bits. One advantage of LTQ_n is that the diameter is only about half of the diameter of Q_n.

1.3 Thesis Organization

This thesis is organized as follows.

In Chapter 2, we introduce basic concepts in graph theory which will be used throughout this thesis.

In Chapter 3, we include the background of our research with a discussion on the relationship between different types of connectivities, showing some known results on transitive graphs and specifically, Cayley graphs.

In Chapter 4, we show that if G is a $\lambda^{(4)}$-connected graph with $\lambda^{(4)}(G) \leq \xi_4(G)$ and the girth $g(G) \geq 8$, and there are not six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3$ $(1 \leq i, j \leq 3)$, then G is maximally 4-restricted edge-connected.

In Chapter 5, we prove that the nature diagnosability of $C\Gamma_n$ under the PMC model and MM* model is $2n-3$ except that, the bubble-sort graph B_4, where $n \geq 4$, and the nature diagnosability of B_4 under the MM* model is 4.

In Chapter 6, we show that the 2-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model and MM* model is $g(n-2)-1$, where $n \geq 4$ and g is the girth of $C\Gamma_n$.

In Chapter 7, we show that the connectivity of CK_n is $\frac{n(n-1)}{2}$, the nature neighbor connectivity of CK_n is n^2-n-2 and the nature diagnosability of CK_n under the PMC model is n^2-n-1 for $n \geq 4$ and under the MM* model is n^2-n-1 for $n \geq 5$.

In Chapter 8, we prove that the nature diagnosability of BS_n is $4n-7$ under the PMC model for $n \geq 4$, the nature diagnosability of BS_n is $4n-7$ under the MM* model for $n \geq 5$.

In Chapter 9, we prove that (1) the connectivity of XQ_n^k is $4n$; (2) the nature connectivity of XQ_n^k is $8n-4$; (3) the nature diagnosability of XQ_n^k under the PMC model and MM* model is $8n-3$ for $n \geq 2$.

In Chapter 10, we show that LTQ_n is tightly $(4n-9)$ super 3-extra connected for $n \geq 6$ and the 3-extra diagnosability of LTQ_n under the PMC model and MM* model is $4n-6$ for $n \geq 5$ and $n \geq 7$, respectively.

In Chapter 11, we prove that diagnosability of $Cay(T_n, S_n)$ is $n-1$ under the comparison diagnosis model for $n \geq 4$.

In Chapter 12, we show the relationship between the g-good-neighbor (extra) diagnosability and g-good-neighbor (extra) connectivity of graphs.

In Chapter 13, we set up a plan for future work.

Chapter 2

Basic Concepts & Preliminary in Graph Theory

In this chapter, we will introduce concepts, definitions and notations which will be used throughout this thesis. Since our research is mainly focused on undirected graphs, thus we will only introduce a few definitions in directed graphs, which mostly for helping defining concepts for undirected graphs. If there is no ambiguity, an undirected graph is called a graph in the thesis. For other concepts, definitions and notations which are not introduced in this chapter, refer to [13].

2.1 Undirected Graphs

A *undirected graph* G is defined as a pair of sets $(V(G), E(G))$, where $V(G)$ is a finite nonempty set of elements called *vertices*, and $E(G)$ is a set (possibly empty) of unordered pairs $\{u, v\}$ called *edges* where vertices $u, v \in V(G)$. For brevity, an edge $\{u, v\}$ is often denoted by uv. $V(G)$ is called the *vertex-set* of G and $E(G)$ is called the *edge-set* of G. A graph G may contain *loops*, that is, edges of the form $\{u, u\}$, and/or *multiple edges*, that is, edges which occur more than once. A *simple graph* is a graph without multiple edges or loops. We denote the number of vertices and edges in G by $v(G)$ and $e(G)$. The *order* of a graph G is the number of vertices in G while the *size* of a graph G is the number of edges in G.

Fig. 2.1 shows an example of a graph of order 7 with vertex-set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and edge-set $\{v_1v_4, v_4v_3, v_3v_5, v_5v_4, v_4v_7\}$. Different from a undirected graph, a *directed graph* D is an ordered triple ($V(D)$, $A(D)$, ψ_D) consisting of a nonempty set $V(D)$ of vertices, a set $A(D)$ of *arcs* together with an *incidence function* ψ_D that associates with each arc of D an ordered pair of (not necessarily distinct) vertices of D.

Let u and v be vertices of a graph G. We say that u is *adjacent* to v if there is an edge e between u and v, that is, $e = uv$. Then we call v a *neighbor* of u. The set of all neighbors of u is called the *neighborhood* of u and is denoted by $N_G(v)$ or $N(v)$ for short if there is no ambiguity. We also say that both vertices u and v are *incident* with edge e, in other words, u and v are the endpoints of e. For example, in Fig. 2.1, vertex v_1 is adjacent to vertex v_4; and vertex v_3 is incident with edges v_3v_4 and v_3v_5.

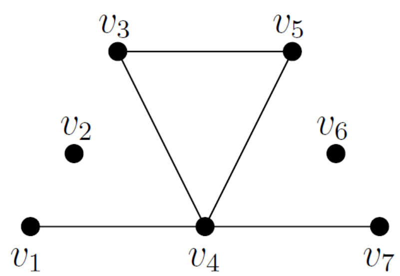


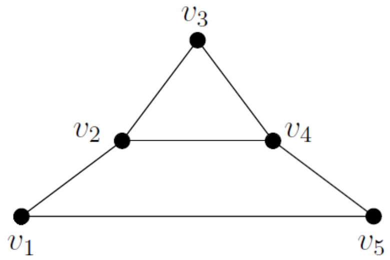
Fig. 2.1 Example of a graph

The *adjacency matrix* of a graph G and vertex-set $V(G) = \{v_1, v_2, ..., v_n\}$ is the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Fig. 2.2 shows a graph of order 5 with its adjacency matrix.

The *degree* of a vertex v of G is the number of vertices adjacent to v, that is, the number of all neighbors of v, which is denoted by $d_G(v)$, $d(v)$ for short if there is no ambiguity. If a vertex v has degree 0, which means that v is not adjacent to any other vertex, then v is called an *isolated vertex*, or *isolate*. A vertex of degree 1 is called an *end vertex*. In Fig. 2.1, the degree of v_4 is 4, v_2 is an isolated vertex, and v_1 is an end vertex. If every vertex of a graph



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Fig. 2.2 Graph and its adjacency matrix

G has the same degree then G is said to be *regular*. For example, the graph in Fig. 2.3 is regular of degree 4.

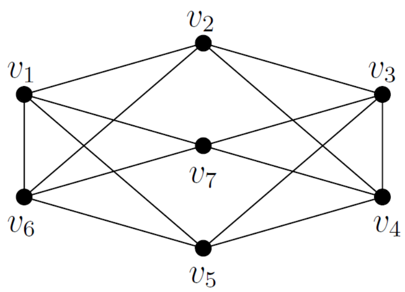


Fig. 2.3 Example of a regular graph

A $v_0 - v_l$ *walk* of a graph G is a finite alternating sequence $v_0, e_1, v_1, e_2, ..., e_l, v_l$ of vertices and edges in G such that $e_i = v_{i-1}v_i$ for each i, $1 \leq i \leq l$. Such a walk may also be denoted by $v_0v_1...v_l$. We note that there may be repetition of vertices and edges in a walk. The *length* of a walk is the number of edges in the walk. A $v_0 - v_l$ walk is *closed* if $v_0 = v_l$. If all the vertices of a $v_0 - v_l$ walk are distinct, then the walk is called a *path*, denoted by $P_k = v_0v_1 \dots v_k$. A *cycle* is a closed path. In Fig. 2.3, $v_1v_2v_6v_7v_4v_2v_3$ is a walk of length 6 which is not a path, $v_1v_2v_3v_4v_5v_6$ is a path of length 5, and $v_1v_7v_3v_5v_1$ is a cycle.

The *distance* from vertex u to v, denoted by $d(u,v)$, is the length of the shortest path from vertex u to vertex v. For example, the distance from vertex v_1 to v_4 of the graph in Fig. 2.3 is 2. The *diameter* of a graph G is the longest distance between any two vertices in G. The *girth* of a graph G is the length of the shortest cycle in G. For example, the graph in Fig. 2.3 has diameter 2 and girth 3.

A graph H is a *subgraph* of G if every vertex of H is a vertex of G, and every edge of H is an edge of G. In other words, $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Let V' be a subset of $V(G)$. The *induced subgraph* $G[V']$ is a subgraph of G consisting of the vertex-set V' together with all the edges uv of G where $u, v \in V'$. In Fig. 2.4, G_1 is an induced subgraph of G, and G_2 is a subgraph of G but not an induced subgraph (because in G_2, $v_7, v_8 \in V(G)$ but there is no edge between v_7 and v_8 while $u_7u_8 \in E(G)$). A *spanning subgraph* of a graph G is a subgraph obtained by edge deletions only, in other words, a subgraph whose vertex set is the entire vertex set of G. If E is the set of deleted edges, this resulting subgraph is denoted by $G \setminus E$ or $G - E$. Observe that every simple graph is a spanning subgraph of a complete graph. If E' is a set of edges, then the *edge-induced subgraph* $G[E']$ is the subgraph of G whose edge set is E' and whose vertex set consists of all end vertices of edges in E'.

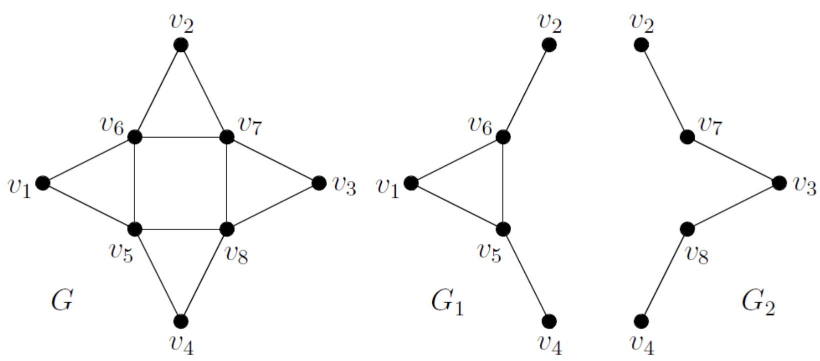


Fig. 2.4 Graph and two of its subgraphs

A *complete graph* on n vertices, denoted K_n, is a graph in which every vertex is adjacent to every other vertex. Thus K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. A graph G is *bipartite* if $V(G)$ can be partitioned into two subsets V_1 and V_2, called *partite sets*, such that there are no edges between any vertices within V_1 and no edges between any vertices within V_2. If G contains all edges joining every vertex in V_1 to every vertex in V_2, then G is called a *complete bipartite graph*. Such a graph is denoted by $K_{m,n}$, where $m = |V_1|$ and $n = |V_2|$. More generally, a *complete n-partite graph* is a graph who has n partite sets $V_1, V_2, \ldots, V_n$ such that two vertices are adjacent if and only if they lie in different partite sets. If $|V_i| = p_i$, then this graph is denoted by $K_{p_1,p_2,\ldots,p_n}$. Fig. 2.5 shows examples of the complete graph K_6 and the complete bipartite graph $K_{3,3}$.

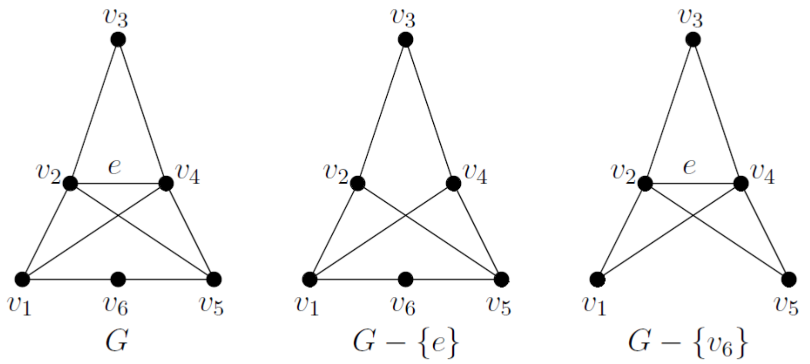


Fig. 2.5 Complete graph K_6 and complete bipartite graph $K_{3,3}$

A graph G is *connected* if for any two distinct vertices u and v of G there is a path between u and v. Otherwise G is *disconnected*. A maximal connected subgraph of G is called a *connected component* or simply a *component* of G. Thus a disconnected graph contains at least two components. For example, the graph in Fig. 2.3 is connected, but the graph in Fig. 2.1 is disconnected (because there is no path between v_2 and any other vertex).

Note that an *acyclic graph* is a graph that contains no cycles. A connected acyclic graph is called a *tree*. A set of acyclic graphs is called *forests*. A vertex of degree 1 is called a *leaf* in tree or forest. A *nontrivial* tree has at least two leaves.

In order for a graph to be connected, there must be at least one path between any two of its vertices.

Let e be an edge of a graph G. Then $G - \{e\}$ is a graph obtained from G by *deleting* the edge e from G. If $G - \{e\}$ is disconnected, then e is called a *bridge*. In general, if E_1 is any set of edges in G then $G - E_1$ is a graph obtained from G by deleting all edges in E_1. Furthermore, E_1 is called an *edge cut* if $G - E_1$ is disconnected.

Similarly, if v is a vertex of a graph G, then $G - \{v\}$ is a graph obtained from G by deleting the vertex v and all edges incident with v. If $G - \{v\}$ is disconnected, then v is called a *cut-vertex*. A graph G is said to be a *non-separable graph* if it does not contain a cut-vertex. Let V_1 be a set of vertices in G. Then $G - V_1$ is a graph obtained from G by deleting all vertices in V_1 and all edges incident with the vertices in V_1. The set V_1 is called a *vertex cut* if $G - V_1$ is disconnected. These concepts are illustrated in Fig. 2.6.

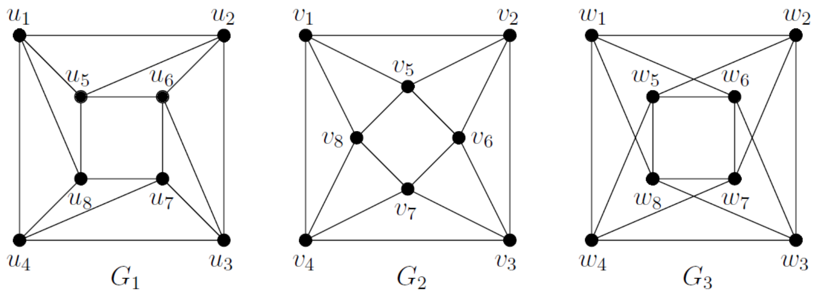


Fig. 2.6 Obtaining new graphs by deleting an edge or a vertex

For two disjoint vertex sets X and Y of $V(G)$, let $[X,Y]$ be the set of edges with one end vertex in X and the other one in Y. A *matching* in a graph is a set of pairwise nonadjacent edges. If M is a matching, the two end vertices of each edge of M are said to be matched under M, and each vertex incident with an edge of M is said to be covered by M. A *perfect matching* is one which covers every vertex of the graph, a *maximum matching* is one which covers as many vertices as possible. A graph is *matchable* if it has a perfect matching.

For a graph $G=(V,E)$, a subset K of V is called a *vertex cover* of G if every edge of E has at least one end vertex in K. A vertex cover of minimum cardinality in G is called *minimum vertex cover*.

Two graphs G_1 and G_2 with n vertices are said to be *isomorphic* if there exists a one-to-one mapping $f : V(G_1) \to V(G_2)$ which preserves all the adjacencies, that is, $f(u)$ and $f(v)$ in G_2 are adjacent if and only if u and v in G_1 are adjacent. In Fig. 2.7, graphs G_1 and G_2 are isomorphic under the mapping $f(u_i) = v_i$, for every $i = 1,2,...,8$. However, graphs G_1 and G_3 are not isomorphic because G_1 contains cycles of length three while G_3 does not and consequently there cannot be any one-to-one mapping preserving adjacencies.

An *automorphism* of a graph G is a one-to-one mapping $f : V(G) \to V(G)$ which preserves all the adjacencies, that is, $f(u)$ and $f(v)$ are adjacent if and only if u and v are. For example, consider the graph G_2 in Fig. 2.7 under the mapping f defined by $f(v_1) = v_3$, $f(v_2) = v_4$, $f(v_3) = v_1$, $f(v_4) = v_2$, $f(v_5) = v_7$, $f(v_6) = v_8$, $f(v_7) = v_5$, $f(v_8) = v_6$. Then f is an automorphism of the graph G_2.

A graph G is *vertex-symmetric* (also known as *vertex-transitive*) if for any two vertices x and y of G, there exists an automorphism of G that carries u to v. For example, all graphs in

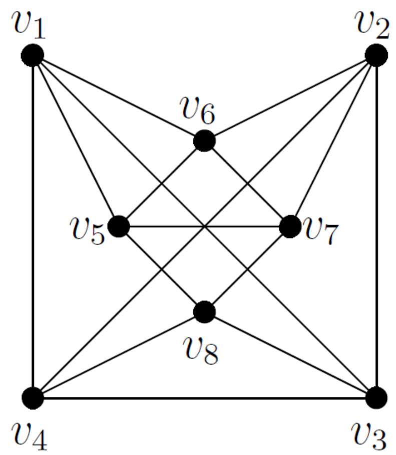


Fig. 2.7 Isomorphism in graphs

Fig. 2.7 are vertex-symmetric, while the graph in Fig. 2.8 is not, because vertex v_6 lies in three cycles of length three, namely, v_1, v_5, v_6; v_2, v_6, v_7; and v_5, v_6, v_7, while vertex v_8 lies in two cycles of length three, namely, v_3, v_4, v_8 and v_5, v_7, v_8. Thus in this case there cannot be an automorphism that carries vertex v_6 to vertex v_8. Similarly, a graph G is *edge-transitive* if given any two edges e_1 and e_2 of G, there is an automorphism of G that maps e_1 to e_2 [9]. In other words, a graph is edge-transitive if its automorphism group acts transitively upon its edges [61].

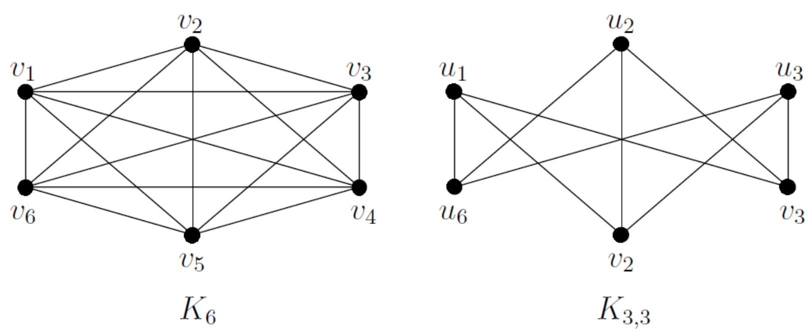


Fig. 2.8 Example of non-vertex-symmetric graph

2.2 Connectivity and Edge Connectivity

Recall that vertex cut of G is a subset V' of V such that $G - V'$ is disconnected. A *k-vertex cut* is a vertex cut of size k. Note that a complete graph has no vertex cut. If G has at least one pair of nonadjacent vertices, the *connectivity* of G, denoted by $\kappa(G)$, is the minimum k for which G has a k-vertex cut, otherwise, we define $\kappa(G)$ to be $\nu - 1$ and $\kappa(G) = 0$ if G is either trivial or disconnected. G is said to be *k-connected* if $\kappa(G) \geq k$. All nontrivial connected graphs are almost 1-connected.

If V' is a minimum vertex cut for G, then the graph can tolerate up to $|V'| - 1$ faulty vertices but cannot tolerate $|V'|$ faulty ones, and so its *fault-tolerance*, denoted by $f(G)$, is equal to $|V'| - 1$, thus $f(G) = \kappa(G) - 1$. The problems of obtaining the vertex-connectivity and fault-tolerance of a graph are equivalent.

Note that an *edge cut* of G is a subset of E of the form $[S, S']$, where S is a nonempty proper subset of V while S' is $V \backslash S$. A *k-edge cut* is an edge cut of size k. If G is nontrivial and E' is an edge cut of G, then $G - E'$ is disconnected. We then define the *edge connectivity* $\lambda'(G)$ of G to be the minimum k for which G has k-edge cut. Let $\lambda'(G) = 0$ if G is either trivial or disconnected. G is said to be *k-edge-connected* if $\lambda'(G) \geq k$. All nontrivial connected graphs are 1-edge-connected.

A *fundamental set* of edge cut sets is set of edge cut sets defined as the following: Given a graph G. We find a spanning tree T of G, then every cut edge of T belongs to one cut set of the fundamental set while every cut set of the fundamental set contains exactly one cut edge of T. It can be shown that each spanning tree uniquely determines a fundamental set [37].

A graph G is *super-connected*, *super-κ* for short (resp. *super-edge-connected*, *super-λ*, for short), if every minimum vertex-cut (resp. edge-cut) isolates a vertex of G [10].

Let $F \subset V(G)(resp.F \subset E(G))$, F is called a *super-vertex-cut* (resp. *super-edge-cut*) of G if $G - F$ is disconnected and every component has at least two vertices. Super vertex-cuts or super-edge-cuts do not always exist. For example, $K_{1,n}$ has no vertex-cuts or super-edge-cuts. The *super-connectivity* (resp. *super-edge-connectivity*) of a graph G, denoted by $\kappa'(G)$ (resp. $\lambda'(G)$), is the minimum cardinally over all super-vertex-cuts (resp. super-edge-cuts) if there is any [107].

A graph G is said to be *hyper-connected* [12], or simply *hyper-κ* (resp. *hyper-edge-connected*, *hyper-λ*, for short), if for every minimum vertex cut F of G (resp. edge-cut), G-F has exactly two components, one of which is an isolated vertex. G is also called *tightly*

$|F|$ *super-connected* in [85] and hence we use these two definitions interchangeably in this thesis.

A subset S of edges in a connected graph G is a k-restricted edge cut if $G-S$ is disconnected and every component of $G-S$ has at least k vertices. The k-restricted edge-connectivity of G, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k-restricted edge cut. A connected graph G is said to be λ_k-connected if G has a k-restricted edge cut. Let $\xi_k(G) = \min\{|[X,\bar{X}]| : |X| = k,\ G[X] \text{ is connected}\}$, where $\bar{X} = V(G)\backslash X$. A graph G is said to be maximally k-restricted edge-connected if $\lambda_k(G) = \xi_k(G)$.

Let F be a set of edges in G. Call F a *cyclic edge-cut* if $G-F$ is disconnected and at least two of its components contain cycles. Clearly, a graph has a cyclic edge-cut if and only if it has two disjoint cycles. We call those graphs which have cyclic edge-cuts *cyclically separable*. The *cyclic edge-connectivity* of G, denoted by $c\lambda(G)$, is defined as follows: if G is not connected, then $c\lambda(G) = 0$; if G is connected but does not have two disjoint cycles, then $c\lambda(G) = \infty$; otherwise, $c\lambda(G)$ is the minimum cardinality over all cyclic edge-cuts of G [69].

Similarly, we can define *cyclic vertex-connectivity* of G, denoted by $\kappa_c(G)$ [111].

The *average connectivity* $\overline{\kappa}(G)$ is defined as the average of the connectivities between all pairs of vertices of G, that is,

$$\overline{\kappa}(G) = \binom{p}{2}^{-1} \sum_{\{u,v\}\subset V} \kappa(u,v)$$

While the (ordinary) connectivity is the minimum number of vertices whose removal separates at least one connected pair of vertices, the average connectivity is a measure for the expected number of vertices that have to be removed to separate randomly chosen pair of vertices.

A graph G is *hamiltonian-connected* if every two vertices of G are connected by a Hamiltonian path [13]. The second smallest eigenvalue of the Laplacian matrix is called the algebraic connectivity [34].

In other words, a *faulty set* of a network is a cut set of the corresponding graph which models the network. For a connected graph $G=(V,E)$, we call a fault set $F \subseteq V$ a *g-good-neighbor faulty set* if $|N(v)\cap(V\backslash F)| \geq g$ for every vertex v in $V\backslash F$. A *g-good-neighbor cut* of a graph G is a g-good-neighbor faulty set F such that $G-F$ is disconnected. The minimum cardinality of g-good-neighbor cuts is defined as the *g-good-neighbor connectivity* or *g-restricted connectivity* of G, denoted by $\kappa^{(g)}(G)$. A connected graph G is said to be *g-good-neighbor connected* or *g-restricted connected* if G has a g-good-neighbor cut. Besides, the 1-good-neighbor connectivity (resp. nature faulty set or faulty cut) is also called *nature connectivity*, denoted by $\kappa^*(G)$ (resp. *nature faulty set or faulty cut*) [67].

A connected graph G is *super-nature-connected* if every minimum nature cut F of $V(G)$ isolates one edge with its two endpoints. Additionally, if $G-F$ has two components, one of which is an edge with its two endpoints, then G is it tightly $|F|$ super-nature-connected.

A fault set $F \subseteq V$ is called a *g-extra faulty set* if every component of $G-F$ has at least $(g+1)$ vertices. A *g-extra cut* of G is a g-extra faulty set F such that $G-F$ is disconnected. The minimum cardinality of g-extra cuts is said to be the *g-extra connectivity* of G, denoted by $\tilde{\kappa}^{(g)}(G)$ [114].

A connected graph G is *super g-extra-connected* if every minimum g-extra cut F of G isolates one connected subgraph of order $g+1$. In addition, if $G-F$ has two components, one of which is the connected subgraph of order $g+1$, then G is *tightly* $|F|$ *super-g-extra-connected.*

2.3 Diagnosability under the PMC Model & MM* Model

The PMC model [59, 112] is a diagnosis model which named after the initials of the three researchers: F.P. Preparata, G. Metze and R.T. Chien. To diagnose a system $G=(V(G),E(G))$, two adjacent nodes in G are capable to perform tests on each other. For two adjacent nodes u and v in $V(G)$, the test performed by u on v is represented by the ordered pair (u,v). The outcome of a test (u,v) is 1 (resp. 0) if u evaluate v as faulty (resp. fault-free). We assume that the test result is reliable (resp. unreliable) if the node u is fault-free (resp. faulty). A test assignment T for G is a collection of tests for every adjacent pair of vertices, which can be modeled as a directed testing graph $T=(V(G),L)$, where $(u,v)\in L$ implies that u and v are adjacent in G. The collection of all test results for a test assignment T is called a *syndrome*. Formally, a syndrome is a function $\sigma: L \mapsto \{0,1\}$.

Recall that the set of all faulty processors in G is called a faulty set in networks. This can be any subset of $V(G)$. For a given syndrome σ, a subset of vertices $F\subseteq V(G)$ is said to be consistent with σ if syndrome σ can be produced from the situation that, for any $(u,v)\in L$ such that $u\in V\setminus F$, $\sigma(u,v)=1$ if and only if $v\in F$. This means that F is a possible set of faulty processors. Since a test outcome produced by a faulty processor is unreliable, a given set F of faulty vertices may produce a lot of different syndromes. On the other hand, different faulty sets may produce the same syndrome. Let $\sigma(F)$ denote the set of all syndromes which F is consistent with. Under the PMC model, two distinct sets F_1 and F_2 in $V(G)$ are said to be *indistinguishable* if $\sigma(F_1)\cap\sigma(F_2)\neq\emptyset$, otherwise, F_1 and F_2 are said to be *distinguishable*. Besides, we say (F_1,F_2) is an *indistinguishable* pair if $\sigma(F_1)\cap\sigma(F_2)\neq\emptyset$; else, (F_1,F_2) is a distinguishable pair.

Using the MM model, which is named after two researchers: J. Maeng and M. Malekth, diagnosis is carried out by sending the same testing task to a pair of processors and comparing their responses. We always assume the output of a comparison performed by a faulty processor is unreliable. In the MM model, a processor sends the same task to a pair of distinct neighbors and then compares their responses to diagnose a system G. The *comparison scheme* of $G=(V(G),E(G))$ is modeled as a multi-graph, denoted by $M=(V(G),L)$, where L is the labeled-edge set. A labeled edge $(u,v)_w\in L$ represents a comparison in which two vertices u and v are compared by a vertex w, which implies $uw,vw\in E(G)$. We usually assume that the testing result is reliable (respectively, unreliable) if the node u is fault-free (respectively, faulty). If $u,v\in F$ and $w\in V(G)\backslash F$, then $(u,v)_w\to 1$. If $u\in F$ and $v,w\in V(G)\backslash F$, then $(u,v)_w\to 1$. If $v\in F$ and $u,w\in V(G)\backslash F$, then $(u,v)_w\to 1$. If $u,v,w\in V(G)\backslash F$, then $(u,v)_w\to 0$. The collection of all comparison results in $M=(V(G),L)$ is called the *syndrome* of the diagnosis, denoted by σ. If the comparison $(u,v)_w$ disagrees, then $\sigma((u,v)_w)=1$. Otherwise, $\sigma((u,v)_w)=0$. Hence, a syndrome is a function from L to $\{0,1\}$. The MM* is a special case of the MM model and each node must test all pairs of its adjacent nodes, i.e., if $uw, vw \in E(G)$, then $(u,v)_w \in L$. For a given syndrome σ, a faulty subset of vertices $F \subseteq V(G)$ is said to be consistent with σ if syndrome σ can be produced from the situation that, for any $(u,v)_w \in L$ such that $w \in V \setminus F$, $\sigma(u,v)_w = 1$ if and only if $u, v \in F$ or $u \in F$ or $v \in F$ under the MM* model. Let $\sigma(F)$ denote the set of all syndromes which F is consistent with. Let F_1 and F_2 be two distinct faulty sets in $V(G)$. If $\sigma(F_1) \cap \sigma(F_2) = \emptyset$, we say (F_1, F_2) is a *distinguishable pair* under the MM* model; else, (F_1, F_2) is an *indistinguishable* pair under the MM* model.

A system $G = (V, E)$ is *g-good-neighbor t-diagnosable* if F_1 and F_2 are distinguishable for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$. The *g-good-neighbor diagnosability* $t_g(G)$ of G is the maximum value of t such that G is g-good-neighbor t-diagnosable.

Proposition 2.3.1 [67] For any given system G, $t_g(G) \leq t_{g'}(G)$ if $g \leq g'$.

In a system $G = (V, E)$, a faulty set $F \subseteq V$ is called a *conditional faulty set* if it does not contain all the neighbor vertices of any vertex in G. A system G is *conditional t-diagnosable* if for every two distinct conditional faulty subsets $F_1, F_2 \subseteq V$ with $|F_1| \leq t, |F_2| \leq t$, F_1 and F_2 are distinguishable. The *conditional diagnosability* $t_c(G)$ of G is the maximum number of t such that G is conditional t-diagnosable. In [44], it was shown that $t_c(G) \geq t(G)$.

Proposition 2.3.2 [83] For a system $G = (V, E)$, $t(G) = t_0(G) \leq t_1(G) \leq t_c(G)$.

In [83], Wang et al. proved that the nature diagnosability of the Bubble-sort graph B_n under the PMC model is $2n-3$ for $n \geq 4$. In [117], Zhou et al. proved the conditional diagnosability of B_n is $4n-11$ for $n \geq 4$ under the PMC model. Therefore, $t_1(B_n) < t_c(B_n)$ when $n \geq 5$ and $t_1(B_n) = t_c(B_n)$ when $n = 4$.

In a system $G = (V, E)$, a faulty set $F \subseteq V$ is called a *g-extra faulty set* if every component of $G - F$ has more than g nodes. G is *g-extra t-diagnosable* if and only if for each pair of distinct faulty g-extra vertex subsets $F_1, F_2 \subseteq V(G)$ such that $|F_i| \leq t$, F_1 and F_2 are distinguishable. The *g-extra diagnosability* of G, denoted by $\tilde{t}_g(G)$, is the maximum value of t such that G is g-extra t-diagnosable.

Proposition 2.3.3 [96] For any given system G, $\tilde{t}_g(G) \le \tilde{t}_{g'}(G)$ if $g \le g'$.

Proposition 2.3.4 [96] For a system G, $t(G) = \tilde{t}_0(G) \le \tilde{t}_g(G) \le t_g(G)$. In particular, $\tilde{t}_1(G) = t_1(G)$.

In [83], Wang et al. studied the nature diagnosability of $C\Gamma_n$ under the PMC model and MM* model and proved that nature diagnosability is less than or equal to the conditional diagnosability of the system. From then on, the 1-good-neighbour diagnosability is also called *nature diagnosability* since it is nature for a fault-free vertex to have at least one fault-free neighbor vertex, comparing with the conditional diagnosability requires that a faulty vertex in faulty set also needs to have at least one fault-free vertex.

2.4 Cayley Graph & Its Basic Properties

Let Q be a finite group, and let S be a generating set of Q such that S has no identity element, where a *finite group* is a mathematical group with a finite number of elements and *a generating set* of a group is a subset such that every element of the group can be expressed as the combination (under the group operation) of finitely many elements of the subset and their inverses. *Directed Cayley graph* $Cay(S,Q)$ is defined as follows: its vertex set is Q, its arc set is $\{(g,gs) : g \in Q, s \in S\}$. Given $t \in S$, we call every arc in $\{(g,gt) : g \in Q\}$ a t-arc. If for each $s \in S$ we also have $s^{-1} \in S$, then for each pair of vertices, there are exactly two arcs of different (opposite) directions. These two arcs between the two vertices can be regarded as one undirected edge and then this Cayley graph is regarded as an *undirected Cayley graph*. We only consider undirected Cayley graph in the thesis.

2.4.1 Circulant Graph

Let $S = \{a_1, a_2, \ldots, a_k\}$ be a set of integers such that $0 < a_1 < \ldots < a_k < (n+1)/2$ and let the vertices of an n-vertex graph be labelled $0, 1, 2, \ldots, n-1$. Then the *circulant graph* $C(n,S)$ has $i \pm a_1, i \pm a_2, \ldots, i \pm a_k$ (mod n) adjacent to each vertex i. The set S is called the *symbol* of $C(n,S)$ [60].

Circulant graphs are Cayley graphs of finite cyclic groups.

Proposition 2.4.1 [63] Every finite cyclic group is isomorphic to an additive group Z_n of residue classes modulo n for some positive integer n.

Therefore, the Cayley graphs generated by finite cyclic groups, namely circulant graphs can be written into $Cay(S', Z_n)$ such that $S' = \{\pm a_1, \pm a_2, \ldots, \pm a_k\}$, where $-a_i = n - a_i$, is equivalent to the circulant graph $C(n,S)$, for $S = \{a_1, \ldots, a_k\}$. Thus the class of Cayley graphs properly contains the class of circulant graphs [60].

2.4.2 Cayley Graph Generated by Transpositions

The *symmetric group* defined over any set is the group whose elements are all the bijections from the set to itself, and whose group operation is the composition of functions. In particular, the finite symmetric group S_n defined over a finite set of n symbols consists of the permutation operations that can be performed on the n symbols, where a *permutation* of a set S is defined as a bijection from S to itself and a *transposition* is a permutation which exchanges two elements and keeps all others fixed. Let S be a set of transpositions in the symmetric group S_n. The *transposition simple graph of S*, denoted by $T(S)$, is defined to be the graph with vertex set $\{1, \ldots, n\}$, and two vertices i and j are adjacent in $T(S)$ whenever $(i, j) \in S$ [36].

Thus, the set S of transpositions in S_n can be represented by the (edge set of the) graph $T(S)$ on n vertices.

Proposition 2.4.2 [36] Let S be a set of transpositions in S_n. Then,
(a) S generates S_n if and only if the transposition simple graph $T(S)$ is connected.
(b) S is a minimal generating set for S_n if and only if the transposition simple graph $T(S)$ is a tree.

Let S be a set of transpositions in S_n. The graph $Cay(S, S_n)$ is called a *Cayley graph generated by transpositions*. If n is even, say $n = 2k$, and $T(S)$ is the graph kK_2 consisting of k independent edges, then the Cayley graph $Cay(S, < S >)$ is isomorphic to the hypercube graph Q_n. Various families of Cayley graphs generated by transpositions have been well-studied and they have specific names [40, 49].

Let S be a set of transpositions in S_n. Let $T(S)$ denote the transposition simple graph of S. If $T(S)$ is the star $K_{1,n-1}$, then $Cay(S,S_n)$ is the *star graph*. If $T(S)$ is the path graph P_n on n vertices, then $Cay(S,S_n)$ is called the *bubble-sort graph*. If $T(S)$ is the cycle graph C_n, then $Cay(S,S_n)$ is called the *modified bubble-sort graph*. If $T(S)$ is $\{(1,i):2 \leq i \leq n\} \cup \{(i,i+1):2 \leq i \leq n-1\}$, then $Cay(S,S_n)$ is the *bubble-sort star graph*. If $T(S)$ is the complete graph K_n, then $Cay(S,S_n)$ is called the *complete transposition graph*. If $T(S)$ is the complete bipartite graph $K_{k,n-k}$, then $Cay(S,S_n)$ is called the *generalized star graph*.

If the transposition simple graph $T(S)$ is a tree, we denote it by Γ_n and the corresponding Cayley graph by $C\Gamma_n$. If the transposition simple graph $T(S)$ is a complete graph K_n, it is also said to be a *nest graph*, denoted by CK_n [87]. If the transposition simple graph $T(S)$ is $\{(1,i):2 \leq i \leq n\} \cup \{(i,i+1):2 \leq i \leq n-1\}$, the bubble-sort star graph is also denoted by BS_n.

Proposition 2.4.3 [91] Let H be a simple connected graph with $n = |V(H)| \geq 3$. If H^1 and H^2 are two different labelled graph obtained by labelling H with $\{1,2,\dots,n\}$, then $Cay(H^1,S_n)$ is isomorphic to $Cay(H^2,S_n)$.

By Theorem 2.4.3, a simple connected graph H can be labelled properly. When $n \geq 4$, $Cay(H,S_n)$ can be decomposed into smaller $Cay(S^*,S_{n-1})$'s as follows, where S^* is a spanning set of S_{n-1}. Given an integer p with $1 \leq p \leq n$, let H_i be the subgraph of $Cay(H,S_n)$ induced by vertices with i in the pth position for $1 \leq i \leq n$. We say $Cay(H,S_n)$ is decomposed along the pth position. When H is a tree T_n, we assume that one vertex of degree one is labelled by n in T_n. If we decompose $Cay(H,S_n)$ along the last position, then H_i and $Cay(T_n - n, S_{n-1})$ are isomorphic. The edges whose end vertices in different H_i's are the cross-edges with respect to the given decomposition. Suppose that the transposition simple graph H is the complete graph K_n. If we decompose $Cay(H,S_n)$ along last position, it is clear to see that H_i and $Cay(H-n,S_{n-1})$ are isomorphic. Besides, we denote $E_{i,j}(G) = E_G(V(H_i),V(H_j))$ for $i,j \in \{1,\dots,n\}$.

Proposition 2.4.4 [1] $\kappa(Cay(T_n, S_n) = n-1$.

Proposition 2.4.5 [1] For any integer $n \geq 1$, $Cay(T_n, S_n)$ is $(n-1)$-regular and vertex-transitive.

2.4.3 Hypercube & k-Ary n-Cube

The *hypercube* Q_n is defined to be the graph on vertex set $\{0,1\}^n$, and two binary strings $x = x_1 \ldots, x_n$ and $y = y_1 \ldots, y_n$ are adjacent vertices in Q_n if and only if they differ in exactly one coordinate. There are other equivalent definitions of the hypercube. Note that the hypercube is isomorphic to the Cayley graph of the permutation group generated by n disjoint transpositions, and so the hypercube graph could have also been defined as a particular kind of Cayley graph.

Let F_2^n be the n-dimensional vector space over the binary field F_2. The set of unit vectors $e_i (i = 1, \ldots, n)$ is a basis for the vector space F_2^n, where a *basis* is a (finite or infinite) set $B = b_i$ of vectors b_i's that spans the whole space and is linearly independent. "Spanning the whole space" means that any vector v can be expressed as a finite sum (called a linear combination) of the basis elements. Note that F_2^n is an abelian group Z_2^n under the operation of vector addition, and the subgroups of Z_2^n correspond to the subspaces of the vector space. Note that the Cayley graph of the abelian group Z_2^n with respect to the set of n unit vectors $e_i (i = 1, \ldots, n)$ is isomorphic to the hypercube graph Q_n. Therefore, we view Q_n as the Cayley graph $Cay(S, Z_2^n)$, where $S = \{e_1, \ldots, e_n\}$ with mod 2.

The hypercube Q_n is an n-regular, vertex-transitive graph on 2^n vertices. For $x, y \in V(Q_n) = Z_2^n$, xy is an edge of Q_n iff $x + y = e_i$ for some i, this edge is said to have edge label (or color) e_i or to be of dimension i. If $y = 1 \ldots 10 \ldots 0$ is a vertex consisting of k 1's and $n-k$ 0's, then the distance from y to the identity vertex $e = 0 \ldots 0$ is exactly k. The path from e to y can be described by a sequence $(e_1, e_2, \ldots, e_k)$ of labels of the edges on the path.

El-Amawy and Latifi [27] proposed the *folded hypercube graph* as a topology for interconnection networks. The folded hypercube graph FQ_n $(n \geq 2)$ is defined to be Cayley graph $Cay(S, Z_2^n)$, where Z_2^n is the abelian group consisting of all 0-1 vectors of length n (with mod

2, componentwise addition) and the generating set $S = e_1, \ldots, en, u$, with $u = e_1 + \ldots, e_n$. In other words, the folded hypercube FQ_n is obtained by taking the hypercube Q_n and adding edges (corresponding to the generator u) which join each vertex to its diametrically opposite vertex.

The *augmented cube* AQ_n is the Cayley graph $Cay(S, Z_2^n)$, where $S = \{e_1, \ldots, e_n\} \cup \{00\ldots00011, 00\ldots00111, 00\ldots01111, \ldots, 11\ldots1111\}$ with mod 2.

k-ary n-cube is defined as the generalization of Hypercube. It is the Cayley graph $Cay(S, Z_k^n)$, where $S = \{\pm e_1, \ldots, \pm e_n\}$ with mod k. Similar to the augmented cube AQ_n, the *augmented* k-ary n-cube $AQ_{n,k}$ is defined as the Cayley graph $Cay(S, Z_k^n)$, where $S = \{\pm e_1, \ldots, \pm e_n\} \cup \{\pm 00\ldots00011, \pm 00\ldots00111, \pm 00\ldots01111, \ldots, \pm 11\ldots1111\}$ with mod k.

The expanded k-ary n-cube, denoted by XQ_n^k ($n \geq 1$ and even $k \geq 6$), is a graph consisting of k^n vertices $\{u_0 u_1 \ldots u_{n-1} : 0 \leq u_i \leq k-1, 0 \leq i \leq n-1\}$. Two vertices $u = u_0 u_1 \ldots u_{n-1}$ and $v = v_0 v_1 \ldots v_{n-1}$ are adjacent if and only if there exists an integer $j \in \{0, 1, \ldots, n-1\}$ such that $u_j = v_j + g$ (mod k) and $u_i = v_i$, for $i \in \{0, 1, \ldots, n-1\} \setminus \{j\}$ and $g \in \{1, -1, 2, -2\}$. For clarity of presentation, we omit writing "(mod k)" if there is no ambiguity. We give two examples as Fig. 2.9 and Fig. 2.10.

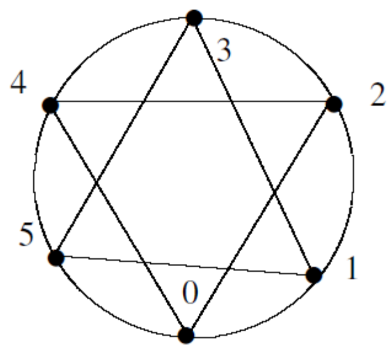


Fig. 2.9 The expanded 6-ary 1-cube XQ_1^6

As shown above, it is straightforward to see that the expanded k-ary n-cube is the generalization of k-ary n-cube and also a Cayley graph $Cay(S, Z_k^n)$, where $S = \{\pm e_1, \ldots, \pm e_n\} \cup \{\pm 2e_1, \ldots, \pm 2e_n\}$ with mod k.

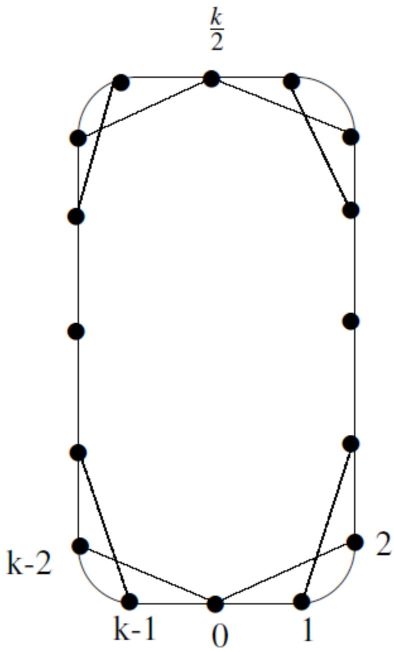


Fig. 2.10 The expanded k-ary 1-cube XQ_1^k

2.5 Hypercube Variants

As was shown by Hillis [42], the hypercube does not have the smallest possible diameter. To achieve smaller diameter with the same number of nodes and links as an n-dimensional cube, a variety of hypercube variants were proposed [22, 41, 77]. Among these variations, Möbius cube, crossed cube, twisted cube, and Mcube have diameters of about half of that of a hypercube of the same size. A common feature of these variants is that the labels of some neighboring nodes may differ in a large number of bits. As a result, certain properties of hypercube are lost in these variants. For example, the design of efficient parallel algorithms on these variants turns out to be a difficult task.

In order to keep as many nice properties of hypercube as possible, a better hypercube variant should be conceptually closer to hypercube. Motivated by this intuition, a new hypercube variant was introduced [41]. The new topology is said to be the *n-dimensional locally twisted cube* LTQ_n because its nodes can be one-to-one labeled with 0-1 binary sequences of length n, so that the labels of any two adjacent nodes differ in at most two successive bits. One advantage of LTQ_n is that the diameter is only about half of the diameter of Q_n.

For an integer $n \geq 1$, a binary string of length n is denoted by $u_1u_2 \ldots u_n$, where $u_i \in \{0,1\}$ for any integer $i \in \{1,2,\ldots,n\}$. The n-dimensional locally twisted cube, denoted by LTQ_n, is an n-regular graph of 2^n vertices and $n2^{n-1}$ edges, which can be recursively defined as follows [109].

For $n \geq 2$, an n-dimensional locally twisted cube, denoted by LTQ_n, is defined recursively as follows: 1). LTQ_2 is a graph consisting of four nodes labeled with 00, 01, 10 and 11, respectively, connected by four edges {00, 01}, {01, 11}, {11, 10} and {10, 00}. 2). For $n \geq 3$, LTQ_n is built from two disjoint copies of LTQ_{n-1} according to the following steps. Let $0LTQ_{n-1}$ denote the graph obtained from one copy of LTQ_{n-1} by prefixing the label of each node with 0. Let $1LTQ_{n-1}$ denote the graph obtained from the other copy of LTQ_{n-1} by prefixing the label of each node with 1. Connect each node $0u_2u_3 \cdots u_n$ of $0LTQ_{n-1}$ to the node $1(u_2+u_n)u_3 \cdots u_n$ of $1LTQ_{n-1}$ with an edge, where "+" represents the modulo 2 addition.

The edges whose end vertices in different $iLTQ_{n-1}s$ are called to be cross-edges. Figs.2.11, Figs.2.12 and Figs.2.13 show four examples of locally twisted cubes. The locally twisted cube can also be equivalently defined in the following non-recursive fashion.

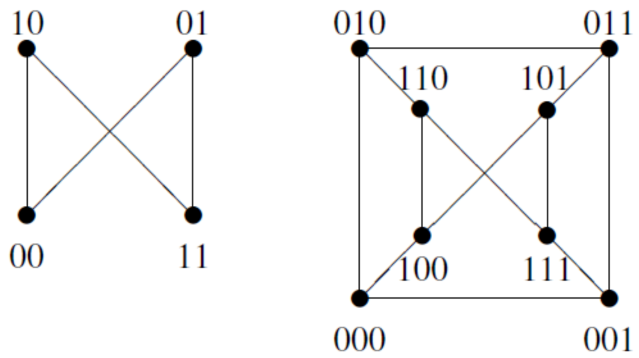


Fig. 2.11 LTQ_2 and LTQ_3

For $n \geq 2$, the n-dimensional locally twisted cube, denoted by LTQ_n, is a graph with $\{0,1\}^n$ as the node set. Two nodes $u_1u_2 \cdots u_n$ and $v_1v_2 \cdots v_n$ of LTQ_n are adjacent if and only if either one of the following conditions are satisfied. 1). $u_i = \overline{v_i}$ and $u_{i+1} = (v_{i+1} + v_n)(mod2)$

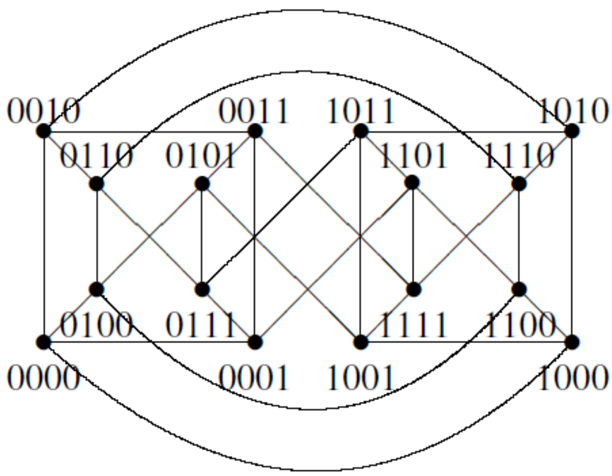
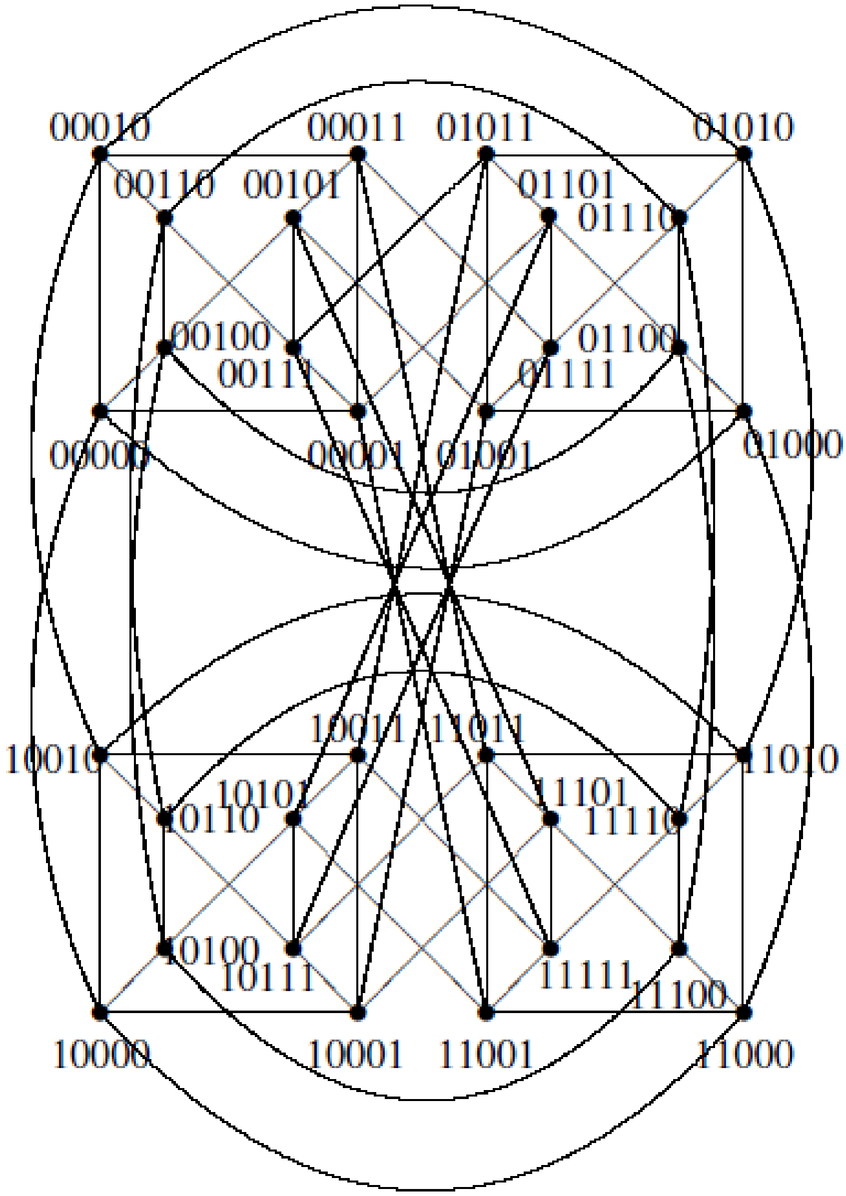


Fig. 2.12 LTQ_4

for some $1 \leq i \leq n-2$, $n \geq 3$ and $u_j = v_j$ for all the remaining bits; 2). $u_i = \overline{v_i}$ for $i \in \{n-1, n\}$, $n \geq 2$ and $u_j = v_j$ for all the remaining bits[109].

Since the labels of any two adjacent nodes differ in at most two successive bits in LTQ_n, it is clear to see that we could not construct one generating set S which determines $N(u)$ and $N(v)$ for any two vertices $u, v \in V(LTQ_n)$, where $n \geq 3$. Therefore, LTQ_n does not belong to Cayley graphs, where $n \geq 3$.


00010
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11010
10101
11101
10110
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10100
10111
11111
11100
10000
10001
11001
11000

Fig. 2.13 LTQ_5

Chapter 3

Connectivities of Cayley Graphs

3.1 Relationship Between Different Types of Connectivities

Connectivity is one of the basic concepts of graph theory, it asks for the minimum number of elements (vertices or edges) that need to be removed to disconnect the graph.

Let v be a vertex in G, where v has the minimum degree $\delta(G)$, if one removes all the vertices adjacent to the vertex v or all edges are incident to v, then we have disconnected v from the rest of the graph G. Thus we know that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

If $\kappa(G) = \delta(G)$ (resp. $\kappa'(G) = \delta(G)$), the G is said to be maximally connected (resp. maximally edge-connected). If $\kappa(G) < \delta(G)$ (resp. $\kappa'(G) < \delta(G)$), then G is not maximally connected (resp. maximally edge-connected), which also means after removing a minimum cut set, each component has at least two vertices.

If $\kappa(G) = \delta(G)$ (resp. $\kappa'(G) = \delta(G)$), it is not necessary that every minimum vertex cut (resp. edge cuts) is a neighbours of a vertex. If every minimum vertex-cut (resp. edge-cut) isolates a vertex of G, which also means every minimum vertex cut (resp. edge cut) is $N(v)$ (resp. $N_e(v)$), G is super-connected, super-κ, for short (resp. super-edge-connected, super-λ, for short).

When the graph is maximally connected, i.e. $\kappa(G) = \kappa'(G) = \delta(G)$, we are then interested in finding a minimum vertex cut (edge cut), whose removal leads to the disconnection of a super-connected graph, where each component has at least two vertices. The cardinality of such conditional vertex cut (resp. edge cut) is said to be super-connectivity $\kappa_s(G)$ (resp. super-edge-connectivity $\lambda_s(G)$) and if G is super-connected (resp. super-edge-connected), we have $\kappa_s(G) > \delta(G)$ (resp.$\lambda_s(G) > \delta(G)$). On the other hand, we observed that if a graph G is maximally connected (resp. maximally edge-connected) but not super-connected (resp. super-edge-connected), by the definition, the minimum vertex cut (resp. edge cut) of size $\delta(G)$ already guarantees that its removal leads to the disconnection of G and each component has at least two vertices and hence $\kappa_s(G) = \delta(G)$ (resp.$\lambda_s(G) = \delta(G)$).

Recall a graph G is hyper-connected, hyper-κ, for short, (resp. hyper-edge-connected, hyper-λ, for short) if every minimum vertex-cut (resp. edge-cut) disconnects G into exactly two components, one of which is an isolated vertex. If a graph G is super-(edge)-connected and there are exactly two components after the removal of minimum vertex (edge) cut, the graph is hyper-(edge)-connected. On the contrary, every hyper-(edge)-connected graph G is super-(edge)-connected.

As a framework, H. Harary [39] has introduced the concept of conditional connectivity by requiring some properties for each component after the removal of a vertex cut or a edge cut. Apart from the super-connectivity (resp. super-edge-connectivity), there are well-known conditional connectivities such as cyclic vertex(edge)-connectivity, g-extra connectivity, g-restricted connectivity and g-good-neighbor connectivity [67, 69, 114].

Firstly, let's see the relationship of maximally (edge)-connectedness, super-(edge)-connectedness, hyper-(edge)-connectedness and cyclic vertex(edge)-connectivity. For a connected graph G with at least two disjoint cycles, if we have $\kappa_c(G) < \delta(G)(c\lambda(G) < \delta(G))$, it is clear that G is not maximally (edge)-connected and hence not super-(edge)-connected. On the contrary, if G is maximally (edge)-connected, then it is straightforward to see that $\kappa_c(G) \geq \delta(G)(c\lambda(G) \geq \delta(G))$. Moreover, if G is super-(edge)-connected(hyper-(edge)-connected), then straightforwardly we have $\kappa_c(G) > \delta(G)(c\lambda(G) > \delta(G))$.

Next, let's see the relationship between super-connectivity (resp. super-edge-connectivity) and restricted connectivity (resp. restricted edge-connectivity). The definition of the two concepts are very similar. The original definitions of these two concepts are as shown in the follow figures.

The purpose of this paper is to study the superconnectivity in a special kind of digraphs: generalized cycles. A (*generalized*) *p-cycle* is a digraph G in which its set of vertices can be partitioned into p parts,

$$V(G) = \bigcup_{\alpha \in \mathbb{Z}_p} V_\alpha,$$

in such a way that the vertices in the partite set V_α are only adjacent to vertices in $V_{\alpha+1}$, where the sum is over $\mathbb{Z}_p$. Observe that any digraph can be shown as a p-cycle with $p = 1$, whereas the bipartite digraphs are generalized 2-cycles.

In order to measure the superconnectivity, we introduce the following concepts. For any nonempty and proper subset of vertices $F \subset V$, consider the set $\Gamma^+(F) = \bigcup_{x \in F} \Gamma^+(x)$, called the *out-neighbourhood* of F. The *positive boundary* and *positive edge-boundary* of F are $\partial^+ F = \Gamma^+(F) \backslash F$ and $\omega^+ F = \{(x, y) \in E\colon\ x \in F, y \in V \backslash F\}$, respectively. The *in-neighbourhood* $\Gamma^-(F)$, *negative boundary* $\partial^- F$ and *negative edge-boundary* $\omega^- F$ are similarly defined. Certainly, if $\bar{F} = V \backslash (F \cup \partial^+ F)$ is nonempty, then $\partial^+ F = \partial^- \bar{F}$ is a disconnecting set. It is also clear that $\omega^+ F = \omega^-(V \backslash F)$ is an edge-disconnecting set. A disconnecting set T is said to be *nontrivial* if, for any vertex $x \notin T$, neither $\Gamma^+(x)$ nor $\Gamma^-(x)$ is contained in T. Notice that if $|\partial^+ F| < \delta$, then $\partial^+ F$ is nontrivial. The nontrivial edge-disconnecting sets are defined in a similar way.

Fig. 3.1 First introduction of super connectivity [7].

A subset $F \subset V(G)$ is said to be nontrivial if it doesn't contain $N(v)$ as its subset for some vertex $v \in V(G)/F$, and a subset $B \subset E(G)$ is said to be nontrivial if it contains no $N_e(v)$ as its subset for some vertex $v \in V(G)$. A nontrivial vertex-set (reps. edge-set) S is called a nontrivial vertex-cut (resp., edge-cut) if $G - S$ disconnected. The super-vertex-connectivity $\kappa_s(G)$ (resp., edge-connectivity $\lambda_s(G)$) of a connected graph G is defined as the minimum cardinality of a nontrivial vertex-cut (resp. edge-cut) if G has a nontrivial vertex-cut (resp., a nontrivial edge-cut), and does not exist otherwise, denoted by ∞ [7, 104]. For the original definition, see Fig. 3.1.

Recall that Esfahanian and Hakimi [28, 29] generalized the notion of connectivity by introducing the concept of the restricted connectivity from the point of view of communication

Formally, the *R-edge-connectivity*, denoted $\lambda(G \mid S: R)$, of a graph $G(V, E)$ is the minimum cardinality $|S|$ of a set S of edges such that $G-S$ is disconnected and S is restricted to a given set R of subsets of E. The *R-vertex-connectivity*, denoted $\kappa(G \mid S: R)$, can be defined similarly with S being a set of vertices and R being a set of subsets of V.

Note that when

$$R=\{X \subset E \mid \text{for any } v \in V,\ I(v) \not\subset X\},$$

then we have

$$\lambda(G:\{\chi \geqslant 2\})=\lambda(G \mid S: R).$$

A similar situation, however, does not arise for the case of vertex-connectivity. That is, in general

$$\kappa(G:\{\chi \geqslant 2\}) \neq \kappa(G \mid S: R),$$

where

$$R=\{X \subset V \mid \text{for any } v \in V,\ A(v) \not\subset X\}.$$

Fig. 3.2 First introduction of restricted connectivity [29].

network. In their paper, they have defined the following: A set $S \subset V(G)$ (resp. $S \subset E(G)$) is called a restricted vertex-set (resp. edge-set) if it contains no $N(x)$ (resp. $N_e(x)$) as its subset for any vertex $x \in V(G)$. A restricted vertex-set (resp., edge-set) S is called a restricted vertex-cut (resp., edge-cut) if $G-S$ is disconnected. The restricted vertex-connectivity (resp., edge-connectivity) of a connected graph G, denoted by $\kappa_r(G)$ (resp., $\lambda_r(G)$), is defined as the minimum cardinality of a restricted vertex-cut (resp., edge-cut) if G has a restricted vertex-cut (resp., edge-cut), and does not exist otherwise.

The four parameters κ_s, κ_r, λ_s and λ_r in conjunction with κ and λ can provide more accurate measurements for fault-tolerance of a large-scale interconnection network. What relationships exist between κ_s and κ_r, λ_s and λ_r?

From definitions, there is no difference between two concepts of nontrivial edge-cuts and restricted edge-cuts, and so $\lambda_s(G)=\lambda_r(G)$ for any graph G provided the edge cuts exist. However, there is a slightly difference between two concepts of nontrivial vertex-cuts S_s and restricted vertex-cuts S_r. We observed that S_s requires S_s contains no $N_(v)$ of vertex $v \in V(G-S)$ while S_r requires S_r contains no $N_(v)$ of vertex $v \in V(G)$, which means $u \notin S_r$ if S_r contains $N(u)$ but $u \in S_s$ if S_s contains $N(u)$. It is clear to obtain the following proposition.

Proposition 3.1.1 [106] Let G be a connected graph, neither $K_{1,n}$ nor K_3. Then

(1) $\kappa_r(G) \geq \kappa_s(G) \geq \kappa(G)$, and if $\kappa_s(G) > \kappa(G) = \delta(G)$ then G is super-connected.

(2) $\lambda_r(G) = \lambda_s(G) \geq \lambda_(G)$, and if $\lambda_s(G) > \lambda(G) = \delta(G)$ then G is super-edge-connected.

Further to the above mentioned connectivity measurements, *g-restricted connectivity* was introduced by Wan and Zhang [81] as the generalization of restricted (vertex) connectivity in 2009. However, g-restricted edge connectivity, which was introduced by Fàbrega and Fiol [31, 32] has a property that each disconnected component contains at least g vertices, while g-restricted connectivity requires the minimum degree of each component is g in 1994. Besides, in 1996, g-extra connectivity, which was introduced by Fàbrega ans Fiol [32] can be seen as another generalization of restricted (vertex) connectivity that requires each component has size at least g+1.

In 2012, Peng et al. [67] proposed a new measure for fault diagnosis of the system, namely, the g-good-neighbor diagnosability (which is also called the g-good-neighbor conditional diagnosability), which requires that every fault-free node has at least g fault-free neighbors. The g-good-neighbor property is in fact equivalent to the g-restricted connectivity used in previous works. In this thesis, we use these two concepts interchangeably. However, here we note that the g-restricted property was introduced in graph theory, where people pay more attention to the static graphs. Thus, the reliability of vertices in these graphs is fixed, i.e., the vertices are totally faulty or fault-free. On the other hand, the term g-good-neighbor is usually used in the area of computer networks, which could be applied to dynamic graphs.

3.2 Connectivity of Symmetric Graphs

In many applications, such as the design of computer networks, it is desired that the network (graph) remains connected even if some of the vertices or links in the network (graph) fail. Recall that the fault-tolerance of a graph G, denoted by $f(G)$, is the maximum number of faults (vertex failures) that can be tolerated without disconnecting the graph. In the definition of this graph parameter, it is assumed that the faulty vertices are chosen by an adversary (this is the worst case scenario).

Edge-transitive graphs and vertex-transitive graphs are excellent candidate for network topology, in particular, their symmetry properties imply that they are maximally (edge)-connected, i.e. highly fault-tolerance.

Mader [58] proved the following result:

Theorem 3.2.1 [58] If G is a connected vertex-transitive graph, then it is maximally edge-connected.

This result settles the question of edge-connectivity for all vertex-transitive graphs and in particular for all Cayley graphs.

Watkins [102] obtained the following sufficient condition for a graph to be maximally connected.

Theorem 3.2.2 [102] If G is a connected edge-transitive graph, then its vertex-connectivity $\kappa(G)$ is equal to its minimum degree $\delta(G)$.

Another sufficient condition for a graph to be maximally connected was obtained by Mader [57]:

Theorem 3.2.3 [57] If G is a connected vertex-transitive graph which does not contain a K_4, then its vertex-connectivity $\kappa(G)$ is equal to its minimum degree $\delta(G)$.

Sufficient conditions for a graph to be maximally connected are given in Theorem 3.2.2 and Theorem 3.2.3. However, there exist graphs which do not satisfy the hypotheses of these assertions and are still maximally connected, for example, some families of circulants and the family of augmented cubes are neither edge-transitive nor K_4-free but are maximally connected.

3.3 Maximally Connected Cayley Graphs

3.3.1 Maximally Edge-Connected Cayley Graphs

It is known from the paper [36] that

Theorem 3.3.1 [36] All Cayley graphs are vertex-transitive.

Combined with Theorem 3.2.1, we know that all connected Cayley graphs are maximally edge-connected and hence $\kappa'(G) = \delta(G)$.

In terms of the maximally vertex-connectedness, if the generating set of a Cayley graph G consists of transpositions, it is clear have that G has no odd cycle. Combined with Theorem 3.2.3, we have G is maximally vertex-connected. Some well-known topology networks (graphs) such Bubble-sort Graphs, Star Graphs are maximally vertex-connected.

3.3.2 Maximally Connected Circulant Graphs, Hypercubes & Generalized Hypercubes

Recall that there exist graphs which do not satisfy the hypotheses of these assertions of Theorem 3.2.2 and Theorem 3.2.3 and which are still maximally connected. Boesch and Tindell [10] characterized the circulants which are maximally connected.

Theorem 3.3.2 [10] The circulants $C(n,S)$, $1 \leq i \leq k$, satisfies $\kappa < \delta$ if and only if for some proper divisor m of n, the number of distinct positive residues modulo m of the numbers $a_1, \ldots, a_k, n-a_k, \ldots, n-a_1$ is less than the minimum of $m-1$ and $\delta m/n$.

Theorem 3.3.3 [36] The hypercube graph Q_n is maximally connected.

As we discussed in chapter 2, the folded hypercube FQ_n is obtained by taking the hypercube Q_n and adding edges (corresponding to the generator u) which join each vertex to its diametrically opposite vertex. The motivation for adding these complementary edges to the hypercube is that they reduce the diameter of the graph from n to about $n/2$. If two vertices in FQ_n differ in more than half of the coordinates, a shorter path between these two vertices is obtained by using the complementary edge. For example, the length of a shortest path in FQ_6 from vertex $e = 000000$ to vertex 011111 is 2; one such shortest path is the path corresponding to the sequence of edge labels (or generators) (u, e_1). The folded hypercube FQ_n is a regular graph of degree $n+1$. Thus, its vertex-connectivity satisfies $\kappa(FQ_n) \leq n+1$.

Theorem 3.3.4 [36] The folded hypercube FQ_n $(n \geq 4)$ is maximally connected.

The *augmented cube* AQ_n is the Cayley graph $Cay(Z_2^n, S)$, where $S = \{e_1, \ldots, e_n\} \cup \{00\ldots00011, 00\ldots00111, 00\ldots01111, \ldots, 11\ldots1111\}$.

Theorem 3.3.5 [23]The augmented cube AQ_n is maximally connected.

The augmented cubes are maximally connected.. However, the augmented cube graphs are neither edge-transitive nor K_4-free.

Theorem 3.3.6 [109] The locally twisted cube LTQ_n is maximally connected.

3.3.3 Maximally Connected Cayley Graphs Generated by Transpositions

In 2009, Ganesan [35] characterized the isomorphism and edge-transitivity of Cayley graphs generated by transpositions:

Theorem 3.3.7 Let S be a set of transpositions generating S_n. The Cayley graph $Cay(S, S_n)$ is edge-transitive if and only if the transposition graph $T(S)$ is edge-transitive.

Theorem 3.3.8 Let S be a set of transpositions generating S_n. Then, the Cayley graph $Cay(S_n, S)$ is edge-transitive if and only if the transposition simple graph $T(S)$ is edge-transitive.

Recall that if a graph is edge-transitive, then it is maximally connected. Thus, combining Theorem 3.3.7 and Theorem 3.3.8, we have the following.

Theorem 3.3.9 [36] Let S be a set of transpositions generating S_n. Then, $Cay(S, S_n)$ is maximally connected.

Since all Cayley graphs generated by transpositions are bipartite, hence are K_4-free. By Theorem 3.2.3, we can also conclude the above theorem that all Cayley graphs generated by transpositions are maximally connected.

3.4 Super-Connected Cayley Graphs

Considering the definitions of maximally (edge)-connected, super-connected and hyper-connected, it is clear that if a Cayley graph G is super-(edge)-connected or hyper-(edge)-connected, G is maximally (edge)-connected.

In 2003, Meng [61] proved the following two theorems:

Theorem 3.4.1 [61] A connected vertex and edge-transitive graph is not super-connected if and only if it is isomorphic to the lexicographic product of a cycle C_n $(n \geq 6)$ or the line graph $L(Q_3)$ of the cube Q_3 by a null graph N_m.

Theorem 3.4.2 [61] A connected vertex and edge-transitive graph G is not hyper-connected if and only if either $G \cong C_n(n \geq 6)$ or $G \cong L(Q_3)$, or there exists a pair of vertices having the same neighbor sets and the number of vertices of G is at least $k+3$, where k is the regularity.

Based on this paper, we know if a maximally vertex-connected Cayley Graph is super-connected or hyper-connected.

3.4.1 Super-(Edge)-Connected Circulant Graphs & Hypercubes

It is known that Cayley graphs are vertex-transitive but not necessarily edge-transitive. It is well known that a circulant graph $C_n(a_1, a_2, \ldots, a_k)$ is connected if and only if $g.c.d.(n, a_1, a_2, \ldots, a_k) = 1$, and the edge connectivity of every connected vertex-transitive graph attains its minimum degree. In [11], Boesch and Wang gave a necessary and sufficient conditions for a circulant graph to be super-edge-connected.

Theorem 3.4.3 [11] A connected circulant is super-edge-connected unless it is $C_p(a)$ or $C_{2n}(2,4,6,\ldots,n-1,n)$ for n odd.

Recursive circulant graphs $G(2^m, 4)$ was proposed by Park and Chwa [80]. This family belongs to the family of circulant graphs denoted by $G(N, d)$ with N, $d \in N$. The vertex set of $G(N, d)$ is $\{0, 1, \ldots, N-1\}$. Two vertices, u and v, are adjacent if and only if $u \pm d^i \equiv v$ (mod N) for some i with $0 \leq i \leq \lceil log_d N \rceil - 1$.

Theorem 3.4.4 [80] $G(2^m, 4)$ is super-connected if and only if $m \neq 2$.

Various networks (graphs) are proposed by twisting some pairs of links in hypercubes. Because of the lack of the unified perspective on these variants, results of one topology are hard to extend to others. To make a unified study of these variants, Vaidya et al. introduced the class of hypercube-like graphs. We denote these graphs as H'-graphs. The class of H'-graphs, consisting of simple, connected, and undirected graphs, contains most of the hypercube variants [45].

Now, we can define the set of n-dimensional H'-graph, H'_n as follows:

1. $H'_1 = \{K_2\}$, where K_2 is the complete graph with two vertices. 2. Assume that G_0, $G_1 \in H'_n$, then $G_0 \oplus G_1$ is graph in H'_{n+1}, where N is any perfect matching between $V(G_0)$ to $V(G_1)$.

Theorem 3.4.5 [45] Every graph in H'_n is both super-connected and super-edge-connected if $n \geq 2$.

3.4.2 Super-Connected Cayley Graphs Generated by Transpositions

As we discussed in chapter 2, we use the transposition simple graph to form the Cayley graphs generated by transpositions.

For the first result, the transposition simple graph is tree.

Theorem 3.4.6 [21] Let G_n be the unidirectional Cayley graph generated by a labelling of a transposition generating tree T_n on n vertices where $n \geq 8$. Then G_n is super-connected.

Then Lemma 3.4.7 is used in the proof of Theorem 3.4.8. For the Cayley graph in Theorem 3.4.8, its transposition simple graph is a normal simple graph with the given restraints.

Lemma 3.4.7 [19] Suppose A is a connected graph with $n \geq 5$ vertices and m edges. If p is the minimum degree of all the non-cut-vertices, then $m \geq max\{n+p-2, 2p-l, 4p-6\}$. Moreover, if $p \geq 3$, then $m \geq 2p$.

Theorem 3.4.8 [19] Suppose G is a Cayley graph obtained from a transposition simple graph S with m edges on $\{1,2,\ldots,n\}$. If $n > 3$, then G is maximally connected. If $n \geq 4$, then G is tightly super-connected.

3.5 Hyper-Connected Cayley Graphs & Other Conditional Connectivities of Cayley Graphs

Graphs should be very well-structured to fulfill the requirements to be hyper-connected. Here we give a theorem to show three hyper-connected Cayley graphs.

Theorem 3.5.1 [55] Let T be a minimal generating set for the symmetric group S_n and let U_n be the set of all transpositions in S_n. Then

(1) for $n \geq 2$, hypercube Q_n is hyper-κ.

(2) for $n \geq 4$, $X = C(S_n, U_n)$ is hyper-κ.

(3) for $n \geq 4$, $X = C(S_n, T)$ is hyper-κ.

Note that $X = C(S_n, U_n)$ is a complete transposition graph and $X = C(S_n, T)$ includes star graph, bubble-sort graph. We have that complete transposition graph, star graph and bubble-sort graph are hyper-κ.

Recall a graph G is super-$\lambda^{(n)}$ if $\lambda^{(m)}(G) = \xi_m(G)$ $(1 \leq m \leq n)$. If a graph G is super-$\lambda^{(n)}$, then the n-restricted edge-connectivity is $\xi_n(G)$. Then we have the following results of n-restricted edge-connectivity for the some families of Cayley graphs.

Theorem 3.5.2 [54] Let $G = G(n; a_1, a_2, \ldots, a_k)$ be a connected circulant with $k \geq 2$ and $a_k < n/2$. Then, the 2-restricted edge-connectivity of G, $\lambda^{(2)}(G) = 4k - 2$.

Theorem 3.5.3 [11] Let $G = C_n(a_1, a_2, \ldots, a_k)$ be a connected circulant graph with $k \geq 2$, then G is super-$\lambda^{(2)}$ if and only if one the three conditions holds:

(1) $a_k < n/2$;

(2) $a_k = n/2$ and $g.c.d.(n, a_1, \ldots, a_{k-1}) = 1$; or

(3) $a_k = n/2$, $g.c.d.(n, a_1, \ldots, a_{k-1}) = 2$ and $n \geq 8k - 8$

Theorem 3.5.4 [62] Star graphs $Cay(S,S_n)$ and hypercubes Q_n are super-$\lambda^{(3)}$ for $n \geq 3$.

Moreover, Meng et al. [62] give the necessary and sufficient conditions that circulant graphs, $G = C_n(a_1, a_2, \ldots, a_k)$ with $n \geq 6$, $k \geq 2$ and $a_k < n/2$ are super-$\lambda^{(3)}$.

We have the following results of n-restricted connectivity and n-extra connectivity for some families of Cayley graphs.

In [105], Xu et al. determined the 1-restricted, 1-extra connectivity and 2-extra connectivity of hypercube Q_n, where $n \geq 3$.

Theorem 3.5.5 [105] $\tilde{\kappa}^{(1)}(Q_n) = \kappa^*(Q_n) = 2n-2$, $n \geq 3$.

Theorem 3.5.6 [105] $\tilde{\kappa}^{(2)}(Q_n) = 3n-5$, $n \geq 4$.

Then, they proved that 1-extra connectivity and 2-extra connectivity of folded hypercube FQ_n are $2n$ for $n \geq 4$ and $4n-4$ for $n \geq 8$, respectively.

Theorem 3.5.7 [105] $\tilde{\kappa}^{(1)}(FQ_n) = \kappa^*(FQ_n) = 2n$, $n \geq 4$.

Theorem 3.5.8 [105] $\tilde{\kappa}^{(2)}(FQ_n) = 4n-4$, $n \geq 8$.

In the end, the g-restricted connectivity of hypercube Q_n is also proved as follow, where $n \geq 3$ and $1 \leq g \leq n-2$.

Theorem 3.5.9 [64, 103] Assume that $n \geq 3$ and $1 \leq g \leq n-2$. Then $\kappa^{(g)}(Q_n) = (n-g)2^g$.

Chapter 4

Sufficient Conditions for Graphs to be Maximally 4-Restricted Edge-Connected

In this chapter, we show that if G is a λ_4-connected graph with $\lambda_4(G) \leq \xi_4(G)$, the girth $g(G) \geq 8$, and there do not exist six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3 \ \ (1 \leq i, j \leq 3)$, then G is maximally 4-restricted edge-connected. The results in this chapter is published in the Australasian Journal of Combinatorics [82].

4.1 Background & Known Results

There is a significant amount of research on k-restricted edge-connectivity [3, 5, 16, 29, 31, 32, 38, 92, 93, 99, 100, 113]. The larger $\lambda_k(G)$ is, the more reliable the network G is [4, 62, 101]. So, we would like the $\lambda_k(G)$ to be as large as possible when design a network topology.

Let's look at the upper bound of $\lambda_k(G)$. For any positive integer k, let $\xi_k(G) = \min\{|[X, \bar{X}]| : |X| = k,\ G[X] \text{ is connected}\}$, where $\bar{X} = V(G) \backslash X$. It has been shown that $\lambda_k(G) \leq \xi_k(G)$ holds for many graphs [6, 14, 65, 115].

Let $G_1, \ldots, G_n$ be n copies of K_t. Add a new vertex u and let u be adjacent to every vertex in $V(G_i)$, $i = 1, \ldots, n$. The resulting graph is denoted by $G^*_{n,t}$. It can be verified that $G^*_{n,t}$

has no $(\delta(G_{n,t}^*)+1)$-restricted edge cuts and $G_{n,t}^*$ is the only exception for the existence of k-restricted edge cuts of a connected graph G when $k \le \delta(G)+1$.

Theorem 4.1.1 [115]. Let G be a connected graph with order at least $2(\delta(G)+1)$ which is not isomorphic to $G_{n,t}^*$ with $t = \delta(G)$. Then for any $k \le \delta(G)+1$, G has k-restricted edge cuts and $\lambda_k(G) \le \xi_k(G)$.

A λ_k-connected graph G is said to be maximally k-restricted edge-connected if $\lambda_k(G) = \xi_k(G)$. When $k = 2$, the k-restricted edge-connectivity of G is the restricted edge-connectivity of G. A maximally k-restricted edge-connected graph is a maximally restricted edge-connected graph. For the research on maximally restricted edge-connected graphs, see [70, 95, 98, 101].

Let G be a λ_k-connected graph and let S be a λ_k-cut of G. In 1989, Plesník and Znám [68] gave the following sufficient condition for a graph to be maximally edge-connected.

Theorem 4.1.2 [68] Let G be a connected graph. If there are not four vertices u_1, u_2, v_1, v_2 in G such that the distance$d(u_i, v_j) \ge 3 \ \ (1 \le i, j \le 2)$, then G is maximally edge-connected.

In 2013, Qin et al. [70] gave the following theorem.

Theorem 4.1.3 [70] Let G be a λ_2-connected graph with the girth $g(G) \ge 4$. If there are not four vertices u_1, u_2, v_1, v_2 in G such that the distance $d(u_i, v_j) \ge 3 \ \ (1 \le i, j \le 2)$, then G is maximally restricted edge-connected.

In 2015, Wang et al. [89] gave the following theorem.

Theorem 4.1.4 [89] Let G be a λ_3-connected graph with the girth $g(G) \ge 5$. If there are not five vertices $u_1, u_2, v_1, v_2, , v_3$ in G such that the distance $d(u_i, v_j) \ge 3 \ \ (1 \le i \le 2; 1 \le j \le 3)$, then G is maximally 3-restricted edge-connected.

In this chapter, we extend the above result to λ_4-connected graph

4.2 Main Results

We firstly give an existing result.

Lemma 4.2.1 [93] Let G be a λ_k-connected graph with $\lambda_k(G) \leq \xi_k(G)$ and let $S = [X,Y]$ be a λ_k-cut of G. If there exists a connected subgraph H of order k in $G[X]$ with the property that

$$\sum_{v \in X \setminus V(H)} |N(v) \cap V(H)| \leq \sum_{v \in X \setminus V(H)} |N(v) \cap Y|,$$

then G is maximally k-restricted edge-connected.

For a λ_4-connected graph, we have,

Theorem 4.2.2 Let G be a λ_4-connected graph with $\lambda_4(G) \leq \xi_4(G)$ and let the girth $g(G) \geq 8$. If there are not six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3$ $(1 \leq i, j \leq 3)$, then G is maximally 4-restricted edge-connected.

Proof: We suppose, on the contrary, that G is not maximally 4-restricted edge-connected. Let $S = [X,Y]$ be a λ_4-cut of G. Denote $X_1 = \{x \in X : N(x) \cap Y \neq \emptyset\}$ and $Y_1 = \{y \in Y : N(y) \cap X \neq \emptyset\}$. Let $X_0 = X \setminus X_1$, $Y_0 = Y \setminus Y_1$, and let $m_0 = |X_0|$, $m_1 = |X_1|$, $n_0 = |Y_0|$ and $n_1 = |Y_1|$. If $|X| = 4$ or $|Y| = 4$, then $\lambda_4(G) \leq \xi_4(G) \leq |S| = \lambda_4(G)$, i.e., G is maximally 4-restricted edge-connected, a contradiction. Therefore $|X| \geq 5$ and $|Y| \geq 5$.

Claim 1. $m_0 \geq 2$ and $n_0 \geq 2$.

We prove this Claim by contradiction. Without loss of generality, assume $m_0 \leq 1$. Let $m_0 = 0$. By the theorems in [76], there is a connected subgraph H of order 4 such that $X_0 \subseteq V(H)$ in $G[X]$. Let $m_0 = 1$ and $X_0 = \{x\}$. Since $G[X]$ is connected, there is a spanning tree T in $G[X]$. Therefore $x \in V(T)$. Since T has two vertices of degree 1, there is a vertex v of degree 1 such that $v \neq x$. Then $T - v$ is a tree and $x \in V(T - v)$. Since there is a vertex v_2 of degree 1 such that $v_2 \neq x$, $T - v - v_2$ is a tree and $x \in V(T - v - v_2)$. Continuing this process, we can obtain a tree T' of order 4 such that $x \in V(T')$. Let $H = (G[X])[V(T')]$. Therefore, in $G[X]$, there is a connected subgraph H of order 4 such that $X_0 \subseteq V(H)$. Let

$u \in X\backslash V(H)$. Then $|[\{u\},Y]| \geq 1$. Since $|V(T')| = 4$, the maximum cardinality of paths is less than or equal to 3. Since $g(G) \geq 8$, $|[\{u\},V(H)]| \leq 1$ holds. Therefore, we have that

$$\begin{aligned}
\sum_{u \in X\backslash V(H)} |N(u) \cap V(H)| &= |[X\backslash V(H), V(H)]| \\
&\leq |X\backslash V(H)| \\
&\leq |[X\backslash V(H), Y]| \\
&= \sum_{u \in X\backslash V(H)} |N(u) \cap Y|. \qquad (4.1)
\end{aligned}$$

By Lemma 4.2.1, G is maximally 4-restricted edge-connected, a contradiction. Therefore $m_0 \geq 2$. Similarly, we have $n_0 \geq 2$. The proof of Claim 1 is completed.

Claim 2. $m_0 = 2$ or $n_0 = 2$.

Suppose that $m_0 \geq 3$ and $n_0 \geq 3$. Then there are six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that $u_1, u_2, u_3 \in X_0$ and $v_1, v_2, v_3 \in Y_0$. By the definition of X_0 and Y_0, we have that $|N(u_i) \cap Y| = 0 = |N(v_j) \cap X|$ for $1 \leq i \leq 3; 1 \leq j \leq 3$. It follows that $d(u_i, v_j) \geq 3 \ \ (i, j \in \{1,2,3\})$, a contradiction. Combining this with Claim 1, we have that $m_0 = 2$ or $n_0 = 2$. The proof of Claim 2 is completed.

Claim 3. In $G[X]$, let H be a connected subgraph of order 4 such that it contains X_0 as most as possible and let $V(H) = \{x_1, x_2, x_3, x_4\}$. If $X_0 = \{u_1, u_2\}$, then

(1) $|X_0 \cap V(H)| = 1$;

(2) $H = u_1x_2x_3x_4$ is a path of length 3, where $u_1 = x_1$, if $u_1 \in V(H)$; and $u_1x_2x_3x_4u_2$ is a path of length 4 in $G[X]$;

(3) $(N(u_1) \cap X)\backslash V(H) = \emptyset$ and $(N(u_2) \cap X)\backslash V(H) = \emptyset$.

Since $|X_0| = 2$, $1 \leq |X_0 \cap V(H)| \leq 2$ holds. We consider the following two cases.

Case 1. $|X_0 \cap V(H)| = 2$.

Since $g(G) \geq 8$, $|[\{u\},V(H)]| \leq 1$ for $u \in X\backslash V(H)$. Note that $X_0 = \{u_1,u_2\} \subseteq V(H)$. Then $|[\{u\},Y]| \geq 1$ for $u \in X\backslash V(H)$. By (2.1), we have that

$$\sum_{u\in X\backslash V(H)} |N(u)\cap V(H)| \leq \sum_{u\in X\backslash V(H)} |N(u)\cap Y|.$$

By Lemma 4.2.1, G is maximally 4-restricted edge-connected, a contradiction.

Case 2. $|X_0 \cap V(H)| = 1$.

In this case, suppose $u_1 \in V(H)$. Since $g(G) \geq 8$, H is a tree of order 4, and $|[\{u\},V(H)]| \leq 1$ for $u \in X\backslash V(H)$. If $|N(u_2)\cap V(H)| = 0$, then $|[\{u\},V(H)]| \leq |[\{u\},Y]|$ for $u \in X\backslash V(H)$. Therefore, we have that

$$\sum_{u\in X\backslash V(H)} |N(u)\cap V(H)| \leq \sum_{u\in X\backslash V(H)} |N(u)\cap Y|.$$

By Lemma 4.2.1, G is maximally 4-restricted edge-connected, a contradiction. Then $|N(u_2)\cap V(H)| = 1$. Suppose that H is not a path. Then H has at least three vertices of degree 1. Let u_2 be adjacent to a vertex y of H. Then there is a vertex v of degree 1 such that $v \neq u_1$ and y in H. Therefore, $(G[X])[V(H-v)\cup\{u_2\}]$ is a connected graph of order 4, a contradiction to the order of H. Then H is a path P of length 3. If u_1 is not a vertex of degree 1, then there is a connected subgraph of order 4 such that it contains u_1,u_2 in $G[V(H)\cup\{u_2\}]$, a contradiction to the order of H. Therefore u_1 is a vertex of degree 1 in P. Let $P = u_1x_2x_3x_4$. Suppose that $N(u_2)\cap V(H) = \emptyset$. Then $|[\{u\},V(H)]| \leq |[\{u\},Y]|$ for $u \in X\backslash V(H)$. Therefore, we have that

$$\sum_{u\in X\backslash V(H)} |N(u)\cap V(H)| \leq \sum_{u\in X\backslash V(H)} |N(u)\cap Y|.$$

By Lemma 4.2.1, G is maximally 4-restricted edge-connected, a contradiction. Therefore, $|N(u_2)\cap V(H)| = 1$. If $N(u_2)\cap\{x_2,x_3\} \neq \emptyset$, a contradiction to the order of H. Then u_2 is adjacent to x_4.

Suppose, on the contrary, that $x \in (N(u_1)\cap X)\backslash V(H)$. Then $P' = xu_1x_2x_3$ is a path of length 3 in $G[X]$. Since $g(G) \geq 8$, $|N(u)\cap V(P')| \leq 1$ for $u \in X\backslash V(P')$. If $N(u_2)\cap V(P') \neq \emptyset$, then there is a connected subgraph H' of order 4 in $G[X]$ with $u_1,u_2 \in V(H')$, a contradiction to that $|X_0 \cap V(H)| = 1$. Therefore, we have that $|N(u_2) \cap V(P')| = 0$ and $|N(u) \cap V(P')| \leq |N(u) \cap Y|$ for $u \in X \backslash V(P')$. Thus,

$$\sum_{u \in X \backslash V(P')} |N(u) \cap V(P')| \leq \sum_{u \in X \backslash V(P')} |N(u) \cap Y|.$$

By Lemma 4.2.1, G is maximally 4-restricted edge-connected, a contradiction. So $(N(u_1) \cap X) \backslash V(H) = \emptyset$ and $d(u_1) = 1$ in $G[X]$.

Suppose, on the contrary, that $x \in (N(u_2) \cap X) \backslash V(H)$. By Claim 3 (2), $P' = x_3x_4u_2x$ is a path of length 3 in $G[X]$. Since $g(G) \geq 8$, $|N(u) \cap V(P')| \leq 1$ for $u \in X \backslash V(P')$. Since $d(u_1) = 1$ in $G[X]$ and $u_1x_2 \in E(G[Y])$, we have $N(u_1) \cap V(P') = \emptyset$. Therefore, we have that $|N(u) \cap V(P')| \leq |N(u) \cap Y|$ for $u \in X \backslash V(P')$. Thus,

$$\sum_{u \in X \backslash V(P')} |N(u) \cap V(P')| \leq \sum_{u \in X \backslash V(P')} |N(u) \cap Y|.$$

By Lemma 4.2.1, G is maximally 4-restricted edge-connected, a contradiction. So $(N(u_2) \cap X) \backslash V(H) = \emptyset$. The proof of Claim 3 is completed.

Similarly to Claim 3, we have that the following claim.

Claim 4. In $G[Y]$, let H^* be a connected subgraph of order 4 such that it contains Y_0 as much as possible and let $V(H^*) = \{y_1, y_2, y_3, y_4\}$. If $Y_0 = \{v_1, v_2\}$, then

(1) $|Y_0 \cap V(H^*)| = 1$;

(2) $H^* = v_1y_2y_3y_4$ is a path of length 3, where $v_1 = y_1$, if $v_1 \in V(H^*)$; and $v_1y_2y_3y_4v_2$ is a path of length 4 in $G[Y]$;

(3) $(N(v_1) \cap Y) \backslash V(H^*) = \emptyset$ and $(N(v_2) \cap Y) \backslash V(H^*) = \emptyset$.

Without loss of generality, suppose $m_0 = 2$. We consider the following cases.

Case 1. $n_0 = 2$.

Claim 5. $|[\{x_2, x_3, x_4\}, \{y_2, y_3, y_4\}]| \leq 1$ in G (See Fig. 4.1).

Suppose $|[\{x_2, x_3, x_4\}, \{y_2, y_3, y_4\}]| \geq 2$. It is sufficient to show that $|[\{x_2, x_3, x_4\}, \{y_2, y_3, y_4\}]| = 2$. Since $x_2x_3x_4$ and $y_2y_3y_4$ are paths, and $|[\{x_2, x_3, x_4\}, \{y_2, y_3, y_4\}]| = 2$, we have that there is a cycle of G whose length is at most 6, a contradiction to $g(G) \geq 8$. The proof of Claim 5 is completed.

Suppose, firstly, that $|[\{x_2,x_3,x_4\},\{y_2,y_3,y_4\}]| = 1$ and $x_{i_0}y_{j_0} \in E(G)$ $(2 \leq i_0 \leq 4, 2 \leq j_0 \leq 4)$. Let $x_i \in \{2,3,4\}\backslash\{i_0\}$ with $x_ix_{i_0} \in E(H)$ and $y_j \in \{2,3,4\}\backslash\{j_0\}$ with $y_jy_{j_0} \in E(H^*)$. By Claim 5, $d(x_i,y_j) \neq 1$. If $d(x_i,y_j) = 2$, then there is a vertex y in $G[Y]$ such that $x_iy, yy_j \in E(G)$ or there is a vertex x in $G[X]$ such that $x_ix, xy_j \in E(G)$. Without loss of generality, suppose that there is a vertex y in $G[Y]$ such that $x_iy, yy_j \in E(G)$. Then there is a cycle C in G, and $x_{i_0}, y_{j_0}, x_i, y_j, y \in V(C)$ and the length of C is 5, a contradiction to $g(G) \geq 8$. Therefore, $d(x_i,y_j) \geq 3$. By Claim 4 (3), $d(x_i,v_i) \geq 3$ for $\{1,2\}$. Similarly to the discussion on x_i, we have that $d(y_j,u_k) \geq 3$ for $k \in \{1,2\}$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_i\}$ and $y \in \{v_1,v_2,y_j\}$, a contradiction.

Suppose, secondly, that $|[\{x_2,x_3,x_4\},\{y_2,y_3,y_4\}]| = 0$. Since there is no $d(x,y) \geq 3$ for every $x \in \{x_2,x_3,x_4\}$ and $y \in \{y_2,y_3,y_4\}$, there are two vertices $x_{i_0} \in \{x_2,x_3,x_4\}$ and $y_{j_0} \in \{y_2,y_3,y_4\}$ such that $d(x_{i_0},y_{j_0}) = 2$. Let $i \in \{2,3,4\}\backslash\{i_0\}$ with $x_ix_{i_0} \in E(H)$ and $j \in \{2,3,4\}\backslash\{j_0\}$ with $y_jy_{j_0} \in E(H^*)$. Since $g(G) \geq 8$, $d(x_i,y_j) \geq 3$ holds. By Claim 4 (3), $d(x_i,v_j) \geq 3$ for $j \in \{1,2\}$. Similarly, $d(y_j,u_i) \geq 3$ for $i \in \{1,2\}$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_i\}$ and $y \in \{v_1,v_2,y_j\}$, a contradiction.

Case 2. $n_0 \geq 3$.

Let $Y_0 = \{y_0,v_1,v_2,v_3,\ldots\}$. By Claim 3 (2), we have that $H = u_1x_2x_3x_4$ and $u_1x_2x_3x_4u_2$ is a path in $G[X]$. Since $g(G) \geq 8$, we have $|N(v) \cap V(H^*)| \leq 1$ for $v \in Y\backslash V(H^*)$. If $|N(y) \cap V(H^*)| = 0$ for $y \in Y_0\backslash V(H^*)$, by Lemma 4.2.1, G is maximally 4-restricted edge-connected, a contradiction. Therefore, there is at least a vertex y_0 in $Y_0\backslash V(H^*)$ such that $|N(y_0) \cap V(H^*)| = 1$.

Case 2.1. $|Y_0 \cap V(H^*)| = 1$.

Let $Y_0 \cap V(H^*) = \{v_1\}$. Note that H^* is a path of length 3 or a $K_{1,3}$. Similarly to the discussion on H, we have that $G[V(H^*) \cup \{y_0\}]$ is a path of length 4, denoted by $P_1 = y_1y_2y_3y_4y_5$, where $v_1 = y_1, y_5 = y_0$. Similarly to Case 1, there is a contradiction.

Case 2.2. $|Y_0 \cap V(H^*)| = 2$.

Let $Y_0 \cap V(H^*) = \{v_1, v_2\}$. Since H^* is a path of length 3 or a $K_{1,3}$, we have that $1 \leq d_{H^*}(v_1, v_2) \leq 3$.

Case 2.2.a. $d_{H^*}(v_1, v_2) = 3$.

In this case, H^* is a path of length 3, denoted by $H^* = y_1y_2y_3y_4$, where $v_1 = y_1, v_2 = y_4$. Similarly to the proof of Claim 5, we have the following claim.

Claim 6. $|[\{x_2, x_3, x_4\}, \{y_2, y_3\}]| \leq 1$ in G (See Fig. 4.2).

Suppose, firstly, that $|[\{x_2, x_3, x_4\}, \{y_2, y_3\}]| = 1$ Without loss of generality, we consider the following cases.

Case 2.2.a.1. $x_2y_2 \in E(G)$.

In this case, $x_3x_2y_2y_3$ is a path in G. Since $g(G) \geq 8$ and Claim 6, $d(x_3, y_3) = 3$ holds. Assume $d(x_3, v_1) = 2$. Since $N(v_1) \cap X = \emptyset$, there is a vertex y in $G[Y]$ such that $x_3y, yv_1 \in E(G)$. Thus, $x_3yv_1y_2x_2x_3$ is a 5-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_3, v_1) = 3$. Similarly, $d(x_3, v_2) \geq 3$. By Claim 3, $d(y_3, u_i) \geq 3$ for $i \in \{1, 2\}$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_3\}$ and $y \in \{v_1, v_2, y_3\}$, a contradiction.

Case 2.2.a.2. $x_3y_2 \in E(G)$.

In this case, $x_2x_3y_2y_3$ is a path in G. By Claim 6, $x_2y_3 \notin E(G)$. If $d(x_2, y_3) = 2$, then there is a vertex y in $G[Y]$ such that $x_2y, yy_3 \in E(G)$ or there is a vertex x in $G[X]$ such that $x_2x, xy_3 \in E(G)$. Without loss of generality, suppose that there is a vertex y in $G[Y]$ such that $x_2y, yy_3 \in E(G)$. Note that $x_3y_2y_3yx_2x_3$ is a 5-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_2, y_3) = 3$. Assume $d(x_2, v_1) = 2$. Since $N(v_1) \cap X = \emptyset$, there is a vertex y in $G[Y]$ such that $x_2y, yv_1 \in E(G)$. Thus, $x_2yv_1y_2x_3x_2$ is a 5-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_2, v_1) = 3$. Assume $d(x_2, v_2) = 2$. Since $N(v_2) \cap X = \emptyset$, there is a vertex y in $G[Y]$ such that $x_2y, yv_2 \in E(G)$. Thus, $x_2yv_2y_3y_2x_3x_2$ is a 6-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_2, v_2) \geq 3$. By Claim 3, $d(y_3, u_i) \geq 3$ for $i \in \{1, 2\}$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_2\}$ and $y \in \{v_1, v_2, y_3\}$, a contradiction.

Suppose, secondly, that $|[\{x_2, x_3, x_4\}, \{y_2, y_3\}]| = 0$. Assume $d(x, y) \geq 3$ for every $x \in \{x_2, x_3, x_4\}$ and $y \in \{y_2, y_3\}$. If $d(x_{i_0}, v_1) = 2$ for $x_{i_0} \in \{x_2, x_3, x_4\}$, then $d(x_i, v_1) \geq 3$ for $i \in \{2, 3, 4\} \backslash \{i_0\}$ by $g(G) \geq 8$. Therefore we have $d(x, y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$

and $y \in \{v_1, y_1, y_2\}$, a contradiction. Then there are two vertices $x_{i_0} \in \{x_2, x_3, x_4\}$ and $y_{j_0} \in \{y_2, y_3\}$ such that $d(x_{i_0}, y_{j_0}) = 2$. Let $i \in \{2,3,4\}\backslash\{i_0\}$ with $x_i x_{i_0} \in E(H)$, and $j \in \{2,3\}\backslash\{j_0\}$ with $y_j y_{j_0} \in E(H^*)$. Since $g(G) \geq 8$, $d(x_i, y_j) \geq 3$ holds. Since $d(x_{i_0}, y_{j_0}) = 2$, $d(x_i, v_j) \geq 3$ for $j \in \{1,2\}$ by $g(G) \geq 8$. By Claim 3, $d(y_j, u_i) \geq 3$ for $i \in \{1,2\}$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1, u_2, x_i\}$ and $y \in \{v_1, v_2, y_j\}$, a contradiction.

Case 2.2.b. $d_{H^*}(v_1, v_2) = 2$.

Suppose, firstly, that $H^* \cong K_{1,3}$, where $V(H^*) = \{v_1, v_2, y_1, y_2\}$ and $d_{H^*}(y_2) = 3$. Since $g(G) \geq 8$, we have $|N(v) \cap V(H^*)| \leq 1$ for $v \in Y\backslash V(H^*)$. If $|N(y) \cap V(H^*)| = 0$ for $y \in Y_0\backslash V(H^*)$, by Lemma 4.2.1, G is maximally 4-restricted edge-connected, a contradiction. Therefore, there is at least a vertex y_0 in $Y_0\backslash V(H^*)$ such that $|N(y_0) \cap V(H^*)| = 1$. If y_0 is adjacent to v_i ($i \in \{1,2\}$), then $(G[Y])[\{v_1, v_2, y_0, y_2\}]$ is a connected subgraph of order 4, a contradiction to the definition of H^*. If y_0 is adjacent to y_2, then $(G[Y])[\{v_1, v_2, y_0, y_2\}]$ is a connected subgraph of order 4, a contradiction to the definition of H^*. Therefore, y_0 is adjacent to y_1 (See Fig. 4.3). Similarly to the proof of Claim 5, we have the following claim.

Claim 7. $|[\{x_2, x_3, x_4\}, \{y_1, y_2\}]| \leq 1$ in G.

Suppose, firstly, that $|[\{x_2, x_3, x_4\}, \{y_1, y_2\}]| = 1$ and $x_{i_0} y_{j_0}$ is an edge in G, where $i_0 \in \{2,3,4\}$ and $j_0 \in \{2,3\}$. Without loss of generality, we consider the following cases.

Case 2.2.b.1. $x_2 y_2 \in E(G)$.

If $d(x_3, v_i) = 2$ for $1 \leq i \leq 2$ or $d(x_3, y_0) = 2$, then there is a vertex y in $G[Y]$ such that $x_3 y, y v_i \in E(G)$ or $x_3 y, y y_0 \in E(G)$. Thus, there is a at most 6-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_3, v_i) \geq 3$ and $d(x_3, y_0) \geq 3$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1, u_2, x_3\}$ and $y \in \{v_1, v_2, y_0\}$, a contradiction.

Case 2.2.b.2. $x_2 y_1 \in E(G)$.

The proof of this case is similar to Case 2.2.b.1.

Case 2.2.b.3. $x_3 y_2 \in E(G)$.

If $d(x_2, v_i) = 2$ for $1 \leq i \leq 2$ or $d(x_2, y_0) = 2$, then there is a vertex y in $G[Y]$ such that $x_2 y, y v_i \in E(G)$ or $x_2 y, y y_0 \in E(G)$. Thus, there is a at most 6-cycle in G, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_2, v_i) \geq 3$ and $d(x_2, y_0) \geq 3$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1, u_2, x_2\}$ and $y \in \{v_1, v_2, y_0\}$, a contradiction.

Case 2.2.b.4. $x_3y_1 \in E(G)$.

The proof of this case is similar to Case 2.2.b.3.

Suppose, secondly, that $|[\{x_2,x_3,x_4\},\{y_1,y_2\}]| = 0$. Assume $d(x,y) \geq 3$ for every $x \in \{x_2,x_3,x_4\}$ and $y \in \{y_1,y_2\}$. If $d(x_{i_0},v_1) = 2$ for $2 \leq i_0 \leq 4$, then $d(x_i,v_1) \geq 3$ for $i \in \{2,3,4\}\backslash\{i_0\}$ by $g(G) \geq 8$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_i\}$ and $y \in \{v_1,y_1,y_2\}$, a contradiction. Then there are two vertices $x_{i_0} \in \{x_2,x_3,x_4\}$ and $y_{j_0} \in \{y_2,y_3\}$ such that $d(x_{i_0},y_{j_0}) = 2$. Let $i \in \{2,3,4\}\backslash\{i_0\}$ with $x_ix_{i_0} \in E(H)$, and $j \in \{2,3\}\backslash\{j_0\}$ with $y_jy_{j_0} \in E(H^*)$. Since $g(G) \geq 8$, $d(x_i,y_j) \geq 3$ holds. Since $d(x_{i_0},y_{j_0}) = 2$, $d(x_i,v_j) \geq 3$ for $j \in \{1,2\}$ by $g(G) \geq 8$. By Claim 3, $d(y_j,u_i) \geq 3$ for $i \in \{1,2\}$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_i\}$ and $y \in \{v_1,v_2,y_j\}$, a contradiction.

Suppose, secondly, that H^* is a path of length 3, denoted $H^* = y_1y_2y_3y_4$. Without loss of generality, suppose $v_1 = y_1, v_2 = y_3$.

Since $g(G) \geq 8$, we have $|N(v) \cap V(H^*)| \leq 1$ for $v \in Y\backslash V(H^*)$. If $|N(y) \cap V(H^*)| = 0$ for $y \in Y_0\backslash V(H^*)$, by Lemma 4.2.1, G is maximally 4-restricted edge-connected, a contradiction. Therefore, there is at least a vertex y_0 in $Y_0\backslash V(H^*)$ such that $|N(y_0) \cap V(H^*)| = 1$. If y_0 is adjacent to v_i ($i \in \{1,2\}$), then $(G[Y])[\{v_1,v_2,y_0,y_2\}]$ is a connected subgraph of order 4, a contradiction to the definition of H^*. If y_0 is adjacent to y_2, then $(G[Y])[\{v_1,v_2,y_0,y_2\}]$ is a connected subgraph of order 4, a contradiction to the definition of H^*. Therefore, y_0 is adjacent to y_4 (See Fig. 4.4). Similarly to the proof of Claim 5, we have the following claim.

Claim 8. $|[\{x_2,x_3,x_4\},\{y_2,y_4\}]| \leq 1$ in G.

Suppose, firstly, that $|[\{x_2,x_3,x_4\},\{y_2,y_4\}]| = 1$ Without loss of generality, we consider the following cases.

Case 2.2.b.5. $x_2y_2 \in E(G)$.

Assume $d(x_3,v_{j_0}) = 2$ for $v_{j_0} \in \{v_1,v_2,y_0\}$. Since $N(v_i) \cap X = \emptyset$ and $N(y_0) \cap X = \emptyset$, there is a vertex y in $G[Y]$ such that $x_3y, yv_i(y_0) \in E(G)$. Thus, there is a cycle C in G whose length of C is at most 7, a contradiction to that $g(G) \geq 8$. Therefore, $d(x_3,v_j) \geq 3$ and $d(x_3,y_0) \geq 3$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_3\}$ and $y \in \{v_1,v_2,y_0\}$, a contradiction.

Case 2.2.b.6. $x_3y_2 \in E(G)$.

Similarly, we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_2\}$ and $y \in \{v_1,v_2,y_0\}$, a contradiction.

Suppose, secondly, that $|[\{x_2,x_3,x_4\},\{y_2,y_4\}]| = 0$.

Assume $d(x,y) \geq 3$ for every $x \in \{x_2,x_3,x_4\}$ and $y \in \{y_2,y_4\}$. Since $g(G) \geq 8$, there is one x_i of x_2,x_3 such that $d(x_i,v_1) \geq 3$. Therefore, by Claim 3, we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_i\}$ and $y \in \{v_1,y_2,y_4\}$, a contradiction. Then there are two vertices $x_{i_0} \in \{x_2,x_3,x_4\}$ and $y_{j_0} \in \{y_2,y_3\}$ such that $d(x_{i_0},y_{j_0}) = 2$. Let $x_{i_0}x_i \in E(H)$. Without loss of generality, we consider the following cases.

Case 2.2.b.7. $d(x_{i_0},y_2) = 2$.

Since $g(G) \geq 8$, $d(x_i,v_j) \geq 3$ for $j \in \{1,2\}$ and $d(x_i,y_4) \geq 3$ hold. Therefore, by Claim 3, we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_i\}$ and $y \in \{v_1,v_2,y_4\}$, a contradiction.

Case 2.2.b.8. $d(x_{i_0},y_4) = 2$.

Similarly, we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_i\}$ and $y \in \{v_2,y_0,y_2\}$, a contradiction.

Case 2.2.c. $d_{H^*}(v_1,v_2) = 1$.

Suppose, firstly, that H^* is a path of length 3, denoted by $P_3 = y_1y_2y_3y_4$. If $v_1 = y_1, v_2 = y_2$, then $N(y_0) \cap V(H^*) = \{y_4\}$. Otherwise, there is a connected subgraph G^* of order 4 in $G[V(H^*) \cup \{y_0\}]$ such that $v_1,v_2,y_0 \in V(G^*)$, a contradiction to the definition of H^*. Since $d_{H^*}(v_2,y_0) = 3$, Similarly to Case 2.2.a, we have that there are six vertices x_1, x_2, x_3, z_1, z_2 and z_3 in G such that the distance $d(x_i,z_j) \geq 3$ $(1 \leq i,j \leq 3)$, a contradiction.

Suppose that $H^* \cong K_{1,3}$, where $d_{H^*}(v_1) = 3$. Then there is a connected subgraph G^* of order 4 in $G[V(H^*) \cup \{y_0\}]$ such that $v_1,v_2,y_0 \in V(G^*)$, a contradiction to the definition of H^*.

Case 2.3. $|Y_0 \cap V(H^*)| = 3$.

Let $Y_0 = \{v_1,v_2,v_3,\ldots\}$. Suppose that $n_0 = 3$. Since $g(G) \geq 8$, $|[\{y\},V(H^*)]| \leq 1$ for $y \in Y \backslash V(H^*)$. Since $Y_0 \subseteq V(H^*)$, we have that

$$\begin{aligned}\sum_{y\in Y\setminus V(H^*)} |N(y)\cap V(H^*)| &= |[Y\setminus V(H^*),V(H^*)]| \\ &\leq |Y\setminus V(H^*)| \\ &\leq |[Y\setminus V(H^*),X]| \\ &= \sum_{y\in Y\setminus V(H^*)} |N(y)\cap X|. \end{aligned} \tag{4.2}$$

By Lemma 4.2.1, G is maximally 4-restricted edge-connected, a contradiction. Then $n_0 \geq 4$. Suppose that $v_1,v_2,v_3 \in Y_0 \cap V(H^*)$. Since H^* is a path of length 3 or a $K_{1,3}$, there is at least a vertex of degree 1 in v_1,v_2,v_3. Without loss of generality, suppose $d_{H^*}(v_1) = 1$ and $v_1 = y_1$.

Case 2.3.1. $H^* = y_1y_2y_3y_4$ is a path of length 3.

Since $|Y_0 \cap V(H^*)| = 3$, we have that $H^* = v_1v_2v_3y_4$ (See Fig. 4.5) or $H^* = v_1v_2y_3v_3$. We consider the following cases.

Case 2.3.1.1. $H^* = v_1v_2v_3y_4$.

Since $g(G) \geq 8$, we have the following claim.

Claim 9. $|[\{x_2,x_3,x_4\},\{y_4\}]| \leq 1$ in G.

Suppose, firstly, that $|[\{x_2,x_3,x_4\},\{y_4\}]| = 1$ and $x_{i_0}y_4 \in E(G)$ for $x_{i_0} \in \{x_2,x_3,x_4\}$. Let $x_ix_{i_0} \in E(H)$. Since $g(G) \geq 8$, we have $d(x_i,v_j) \geq 3$ for $j \in \{1,2,3\}$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_i\}$ and $y \in \{v_1,v_2,v_3\}$, a contradiction.

Suppose, secondly, that $|[\{x_2,x_3,x_4\},\{y_4\}]| = 0$.

Since there is no $d(x_i,v_j) \geq 3$ for every $i \in \{2,3,4\}$ and every $j \in \{1,2,3\}$, there is one $d(x_{i_0},v_{j_0}) = 2$ for $i_0 \in \{2,3,4\}$ and $j_0 \in \{1,2,3\}$. Let $x_ix_{i_0} \in E(H)$. Since $g(G) \geq 8$, $d(x_i,v_j) \geq 3$ for every $j \in \{1,2,3\}$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1,u_2,x_i\}$ and $y \in \{v_1,v_2,v_3\}$, a contradiction.

Case 2.3.1.2. $H^* = v_1v_2y_3v_3$.

Similarly to Case 2.3.1.1, we have that there are six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i,v_j) \geq 3$ $(1 \leq i,j \leq 3)$, a contradiction.

Case 2.3.2. $H^* \cong K_{1,3}$.

Let $d(y_2) = 3$ in H^*. Since $|Y_0 \cap V(H^*)| = 3$, we have that $y_2 = v_2$ and $y_2 \neq v_2$ or v_3 or v_3. Similarly to Case 2.3.1, we have that there are six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3$ $(1 \leq i, j \leq 3)$, a contradiction.

Case 2.4. $|Y_0 \cap V(H^*)| \geq 4$.

If $d(x_i, v_j) \geq 3$ for every $i \in \{2,3,4\}$ and every $j \in \{1,2,3,4\}$, then there are six vertices u_1, u_2, x_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3$ $(i, j \in \{1,2,3\})$, a contradiction. Then $d(x_{i_0}, v_{j_0}) = 2$ for $i_0 \in \{2,3,4\}$ and $j_0 \in \{1,2,3,4\}$. Since $g(G) \geq 8$, $d(x_{i_0}, v_j) \geq 3$ for every $j \in \{1,2,3,4\} \backslash \{j_0\}$. Therefore we have $d(x,y) \geq 3$ for every $x \in \{u_1, u_2, x_{i_0}\}$ and $y \in \{v_j : j \in \{1,2,3,4\} \backslash \{j_0\}\}$, a contradiction.

From Cases 1 and 2, we have that G is maximally 4-restricted edge-connected. □

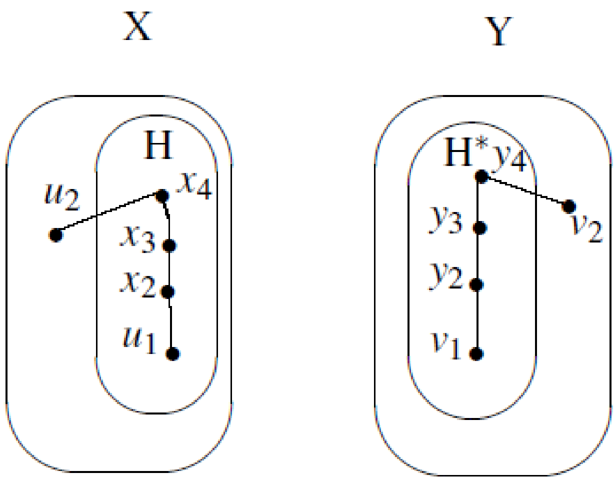

Fig. 4.1 The structure of $G[X]$ and $G[Y]$ in Claim 5 of Theorem 4.2.2

4.3 Conclusion

In this chapter, we showed a sufficient condition for graphs to be maximally 4-restricted edge-connected, i.e., if G is a λ_4-connected graph with $\lambda_4(G) \leq \xi_4(G)$ and the girth $g(G) \geq 8$, and there are not six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3$ for $1 \leq i, j \leq 3$, then G is maximally 4-restricted edge-connected. Our future work along this direction is to investigate the problem of the maximally k-restricted edge-connected graph.

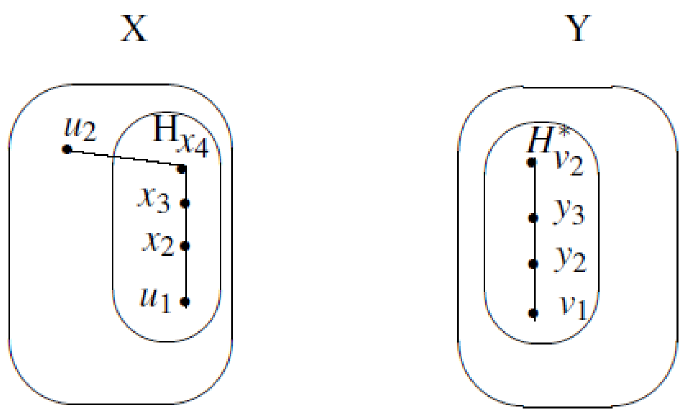

Fig. 4.2 The structure of $G[X]$ and $G[Y]$ in Claim 6 of Theorem 4.2.2

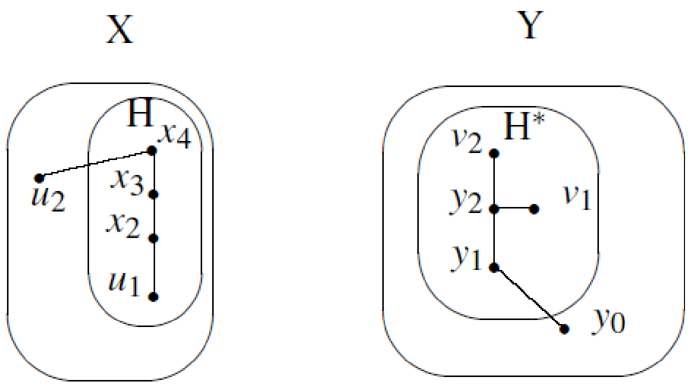

Fig. 4.3 The structure of $G[X]$ and $G[Y]$ in Case 2.2.b of Theorem 4.2.2

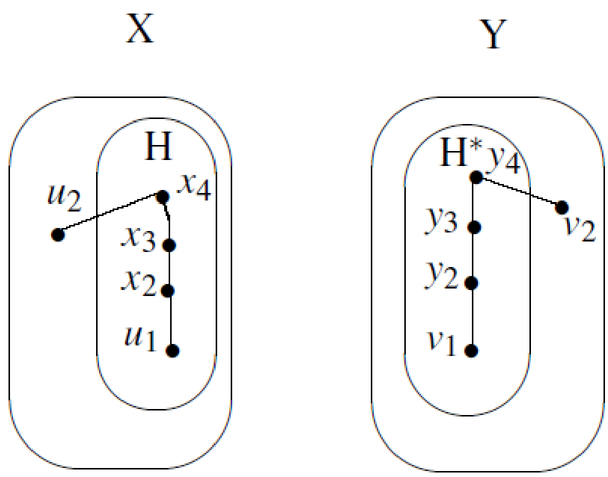

Fig. 4.4 The structure of $G[X]$ and $G[Y]$ in Case 2.2.b.4 of Theorem 4.2.2

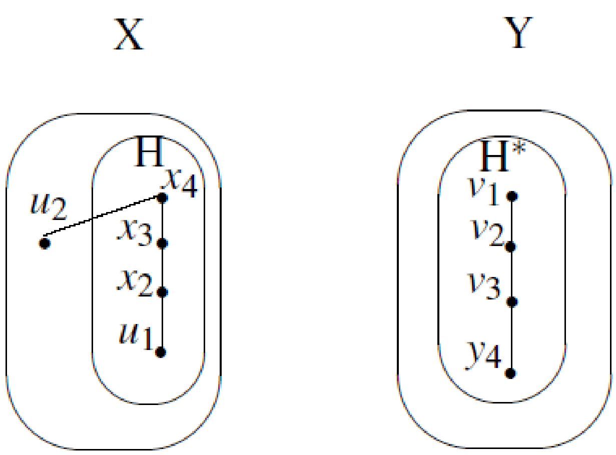

Fig. 4.5 The structure of $G[X]$ and $G[Y]$ in Case 2.3.1 of Theorem 4.2.2

Chapter 5

Nature Diagnosability of Cayley Graphs Generated by Transposition Trees under the PMC Model & MM* Model

In this chapter, we show that the nature diagnosability of $C\Gamma_n$ under the PMC model and MM* model is $2n-3$ except the Bubble-sort graph B_4 under MM* model, where $n \geq 4$, and the nature diagnosability of B_4 under the MM* model is 4. The results presented in this chapter is published in International Journal of Computer Mathematics [83].

5.1 Cayley Graphs Generated by Transposition Trees

In chapter 2, we give the definition of $C\Gamma_n$ and it is easy to see that $C\Gamma_n$ is a $(n-1)$-regular graph on $n!$ vertices. Recently $C\Gamma_n$ as an interconnection network model received much attention, see [19–21, 48, 53, 78, 81, 108] for details.

From the definition of $C\Gamma_n$ and the basic property of Cayley graphs as discussed in chapter 3, we have the following theorem.

Theorem 5.1.1 ([108]) $C\Gamma_n$ is vertex-transitive and bipartite.

Here we give a theorem in Group Theory together with Theorem 5.1.1 for proving the Proposition 5.1.1.

Theorem 5.1.2 ([46]) Every non-identity permutation in the symmetric group is uniquely (up to the order of the factors) a product of disjoint cycles, each of which has length at least 2.

As we defined in chapter 2, Star graphs are Cayley graphs $C\Gamma_n$ generated by transposition tree. For every two transpositions in the transposition simple graph of a Star graph, they are not disjoint, it is obvious that the girth of Star graph is 4. However, there exists 4-cycle in $C\Gamma_n$ if there exists one pair of disjoint transpositions in the corresponding transposition simple graph. This leads to the difference of the results in the following two Propositions.

Proposition 5.1.1 ([116]) Let $C\Gamma_n$ be the Star graph. If two vertices are adjacent, there is no common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| = 0$. If two vertices are not adjacent, there is at most one common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| \leq 1$.

Proposition 5.1.2 If $C\Gamma_n$ is not Star graph and two vertices u, v are adjacent, there is no common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| = 0$. If two vertices u, v are not adjacent, there are at most two common neighbors vertex of these two vertices, i.e., $|N(u) \cap N(v)| \leq 2$.

Proof: In this proof, a permutation is denoted by a product of disjoint cycles. The two cases can be proved by contradiction. For case (1), if two vertices are adjacent and they have a common neighbor vertex, these 3 vertices will form a cycle of length 3. It is a contradiction to Theorem 5.1.1 that there are no odd cycles in a bipartite graph $C\Gamma_n$. For case (2), let two vertices be not adjacent. Suppose, on the contrary, that $|N(u) \cap N(v)| \geq 3$. By Theorem 5.1.1, without loss of generality, assume that $u = (1)$, i.e., u is the identity vertex. Then $v \notin E(\Gamma_n)$. It is sufficient to suppose that $\{(ia), (jb), (kc)\} \subseteq E(\Gamma_n)$, $\{(ia), (jb), (kc)\} \subseteq N(u) \cap N(v)$ and $|\{(ia), (jb), (kc)\}| = 3$. Since $C\Gamma_n$ is not the Star graph, the girth of $C\Gamma_n$ is 4. Since $u, (ia), v, (jb), u$ is a cycle of length 4, we have that $v = (ia)(jb)$ and (ia) is disjoint to (jb).

Since $u,(ia),v,(kc),u$ is also a cycle of length 4, we have that $v=(ia)(kc)$ and (ia) is disjoint to (kc). By Theorem 5.1.2, $v=(ia)(jb)=(ia)(kc)$. Thus, $(jb)=(kc)$, a contradiction to the fact that $|\{(ia),(jb),(kc)\}|=3$. Therefore, $|N(u)\cap N(v)|\leq 2$. The proof is complete. □

In order to determine the R_1-connectivity of $C\Gamma_n$, we divide $C\Gamma_n$ into two parts, one is a Star graph and the other is not a Star graph.

Lemma 5.1.3 ([81]) For $n\geq 3$, if $C\Gamma_n$ is a Star graph, then the R_1-connectivity of $C\Gamma_n$ is $2n-4$, i.e., $\kappa^*(C\Gamma_n)=2n-4$.

Since the R_1-connectivity of Star graph is already determined by Lemma 5.1.3, we only need to determine the R_1-connectivity of $C\Gamma_n$ by using Proposition 5.1.2, where $C\Gamma_n$ is not a Star graph.

Lemma 5.1.4 For $n\geq 3$, the R_1-connectivity of $C\Gamma_n$ is $2n-4$, i.e., $\kappa^*(C\Gamma_n)=2n-4$.

Proof: By Lemma 5.1.3, $\kappa^*(C\Gamma_n)=2n-4$ if $C\Gamma_n$ is a Star graph. Thus, suppose that $C\Gamma_n$ is not a Star graph and $n\geq 4$. Then the girth of $C\Gamma_n$ is 4. By Theorem 2.4.3, let $(12)\in E(\Gamma_n)$ and $A=\{(1),(12)\}$. Then $C\Gamma_n[A]=K_2$. Since $C\Gamma_n$ has no 3-cycles, $|N_{C\Gamma_n}(A)|=2n-4$. Let $F_1=N_{C\Gamma_n}(A)$ and $F_2=A\cup N_{C\Gamma_n}(A)$.

In F_1, we find at most two vertices adjacent to one vertex x in $S_n\setminus F_2$.

Claim 1. For any $x\in S_n\setminus F_2$, $|N_{C\Gamma_n}(x)\cap F_2)|\leq 2$.

Let $(ki),(lj)\in E(\Gamma_n)$, where $3\leq i,j\leq n$. Since $C\Gamma_n$ is a bipartite graph, there is no 5-cycle $(1),(ki),x,(12)(lj),(12),(1)$ of $C\Gamma_n$. Note that $C\Gamma_n-F_1$ has two parts $C\Gamma_n-F_2$ and $C\Gamma_2$ (for convenience). Since $F_1=N_{C\Gamma_n}(A)$, x is not adjacent to each of $V(C\Gamma_2)=A$. By Proposition 5.1.2, $|N_{C\Gamma_n}(x)\cap F_2)|\leq 2$ for any $x\in S_n\setminus F_2$.

By Claim 1, $\delta(C\Gamma_n-F_2)\geq n-1-2=n-3\geq 1$, since $n\geq 4$ by assumption. $C\Gamma_n-F_1$ has two parts $C\Gamma_n-F_2$ and $C\Gamma_2=K_2$ (for convenience). Note that $\delta(C\Gamma_2)=1$. Therefore, $\delta(C\Gamma_n-F_1)\geq 1$ for $n\geq 4$. Thus, F_1 is a nature cut. Thus, $\kappa^1(C\Gamma_n)\leq 2n-4$.

Let F be a subset of S_n such that $|F|\leq 2n-5$.

Claim 2. F is not a nature cut of $C\Gamma_n$.

This Claim is shown by induction on n. $|F| \leq 2n-5 = 2\times 3-5 = 1$ when $n=3$. Since $C\Gamma_3$ is a 6-cycle, $C\Gamma_3 - F$ is connected. Assume that $C\Gamma_{n-1} - F$ is connected when $|F| \leq 2(n-1)-5$. Let $|F| \leq 2n-5$ in the following paragraphs.

By Theorem 2.4.3, a transposition tree Γ_n can be labelled properly. We assume that one vertex of degree one is labelled by n in Γ_n, where $n \geq 4$. We decompose $Cay(\Gamma_n, S_n)$ along the last position, denoted by H_i $(i = 1,2,\ldots,n)$. Then H_i and $Cay(\Gamma_n - n, S_{n-1})$ are isomorphic. The edges whose end vertices in different H_i's are the cross-edges with respect to the given decomposition. We note that each vertex is incident to exactly one cross-edge and there are $(n-2)!$ independent cross-edges between two different H_i's. Let $F_i = F \cap V(H_i)$. We consider the following cases.

Case 1. $|F_i| \leq 2(n-1)-5 = 2n-7$.

In this case, $H_i - F_i$ is connected. Since there are $(n-2)!$ independent cross-edges between two different H_i's and $(n-2)! > 2n-7$ as $n \geq 4$, $C\Gamma_n - F$ is connected.

Case 2. $|F_1| = 2n-6$.

In this case, $|F_i| \leq 1$ $(i = 2,3,\ldots,n)$. Since $|F_1| = 2n-6$, let F_1 be a nature cut of H_1. Since each component of $H_1 - F_1$ has at least two vertices and each of them has two outside neighbors, there is a vertex adjacent to one of H_i $(i = 2,3,\ldots,n)$. Therefore, $C\Gamma_n - F$ is connected.

Case 3. $|F_1| = 2n-5$.

In this case, $|F_i| = 0$ $(i = 2,3,\ldots,n)$. Each vertex of $H_1 - F_1$ is adjacent to one of H_i $(i = 2,3,\ldots,n)$. Therefore, $C\Gamma_n - F$ is connected. The proof of Claim 2 is completed. Therefore, $\kappa^*(C\Gamma_n) = 2n-4$. □

Now we determined the R_1-connectivity of $C\Gamma_n$, which is a indispensable part in proof to determine the nature diagnosability of $C\Gamma_n$ under PMC Model or MM*, where $n \geq 3$.

5.2 The Nature Diagnosability of Cayley Graphs Generated by Transpositions Trees under the PMC Model

In this section, we will obtain the nature diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n under the PMC model.

Firstly we give the necessary and sufficient condition of that a system (graph) G is g-good-neighbor t-diagnosable under PMC model.

Theorem 5.2.1 ([112]) A system $G=(V,E)$ is g-good-neighbor t-diagnosable under the PMC model if and only if there is an edge $uv \in E$ with $u \in V \setminus (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of V with $|F_1| \le t$ and $|F_2| \le t$ (See Fig. 5.4).

Secondly, as we defined the nature faulty set and nature diagnosability and in chapter 2, it is straightforward to obtain the following theorem.

Theorem 5.2.2 A system $G=(V,E)$ is nature t-diagnosable under the PMC model if and only if there is an edge $uv \in E$ with $u \in V \setminus (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of nature faulty subsets F_1 and F_2 of V with $|F_1| \le t$ and $|F_2| \le t$ (See Fig. 5.4).

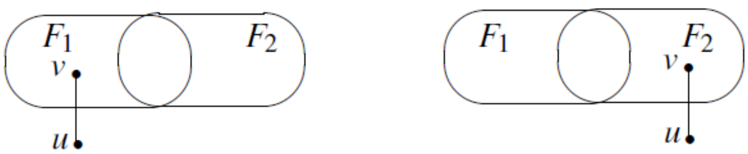

Fig. 5.1 Illustration of a distinguishable pair (F_1, F_2) under the PMC model

Secondly, we derive an important lemma which will be used in the proof to determine the nature diagnosability of $C\Gamma_n$ under PMC Model, where $n \ge 4$.

Lemma 5.2.3 Let $A=\{(1),(12)\}$ and $C\Gamma_n$ be defined as above. If $n \ge 4$, $F_1 = N_{C\Gamma_n}(A)$, $F_2 = A \cup N_{C\Gamma_n}(A)$, then $|F_1| = 2n-4$, $|F_2| = 2n-2$, $\delta(C\Gamma_n - F_1) \ge 1$, and $\delta(C\Gamma_n - F_2) \ge 1$.

Proof: By $A = \{(1),(12)\}$, we have $C\Gamma_n[A] \cong C\Gamma_2 = K_2$. Since $C\Gamma_n$ has no 3-cycles, $|N_{C\Gamma_n}(A)| = 2n-4$. Thus from the calculation, we have $|F_1| = 2n-4$, $|F_2| = |A|+|F_1| = 2n-2$.

In F_1, we will show that at most two vertices adjacent to one vertex x in $S_n \setminus F_2$, i.e., $|N_{C\Gamma_n}(x) \cap F_2)| \leq 2$ for any $x \in S_n \setminus F_2$. Note that $C\Gamma_n - F_1$ has two parts $C\Gamma_n - F_2$ and $C\Gamma_2$ (for convenience). Since $F_1 = N_{C\Gamma_n}(A)$, x is not adjacent to each of $V(C\Gamma_2) = A$. Suppose that the girth of $C\Gamma_n$ is 6. Then $C\Gamma_n$ is a Star graph. By Proposition 5.1.1, $|N_{C\Gamma_n}(x) \cap F_2)| \leq 1$ for any $x \in S_n \setminus F_2$. Suppose that the girth of $C\Gamma_n$ is 4. Then $C\Gamma_n$ is not a Star graph. By Proposition 5.1.2, $|N_{C\Gamma_n}(x) \cap F_2)| \leq 2$ for any $x \in S_n \setminus F_2$. Therefore, $\delta(C\Gamma_n - F_2) \geq n-1-2 = n-3$. $C\Gamma_n - F_1$ has two parts $C\Gamma_n - F_2$ and $C\Gamma_2$ (for convenience). Note that $\delta(C\Gamma_2) = 1$. Since $n \geq 4$, $\delta(C\Gamma_n - F_2) \geq n-3 \geq 1$. Therefore, $\delta(C\Gamma_n - F_1) \geq 1$ for $n \geq 4$. □

Lemma 5.2.4 A graph of minimum degree 1 has at least two vertices.

The proof of Lemma 5.2.4 is straightforward.

Lemma 5.2.5 Let $n \geq 4$. Then the nature diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n under the PMC model $t_1(C\Gamma_n) \leq 2n-3$.

Proof: Let A be defined as above, and let $F_1 = N_{C\Gamma_n}(A)$, $F_2 = A \cup N_{C\Gamma_n}(A)$ (See Fig. 5.2). By Lemma 5.2.3, $|F_1| = 2n-4$, $|F_2| = 2n-2$, $\delta(C\Gamma_n - F_1) \geq 1$ and $\delta(C\Gamma_n - F_2) \geq 1$. Therefore, F_1 and F_2 are both nature faulty sets of $C\Gamma_n$ with $|F_1| = 2n-4$ and $|F_2| = 2n-2$. Since $A = F_1 \triangle F_2$ and $N_{C\Gamma_n}(A) = F_1 \subset F_2$, there is no edge of $C\Gamma_n$ between $V(C\Gamma_n) \backslash (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 5.2.2, we can deduce that $C\Gamma_n$ is not nature $(2n-2)$-diagnosable under PMC model. Hence, by the definition of nature diagnosability, we conclude that the nature diagnosability of $C\Gamma_n$ is less than $2n-2$, i.e., $t_1(C\Gamma_n) \leq 2n-3$. □

Lemma 5.2.6 Let $n \geq 4$, the nature diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n under the PMC model $t_1(C\Gamma_n) \geq 2n-3$.

Proof: By the definition of nature diagnosability, it is sufficient to show that $C\Gamma_n$ is nature $(2n-3)$-diagnosable. By Theorem 5.2.2, to prove $C\Gamma_n$ is nature $(2n-3)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(C\Gamma_n)$ with $u \in V(C\Gamma_n) \backslash (F_1 \cup F_2)$

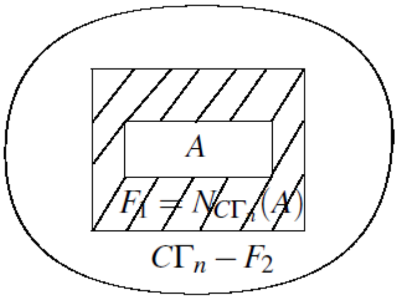

Fig. 5.2 An illustration about the proofs of Lemma 5.2.5 and 5.3.3.

and $v \in F_1 \bigtriangleup F_2$ for each distinct pair of nature faulty subsets F_1 and F_2 of $V(C\Gamma_n)$ with $|F_1| \le 2n-3$ and $|F_2| \le 2n-3$.

We prove this claim by contradiction. Suppose that there are two distinct nature faulty subsets F_1 and F_2 of $V(C\Gamma_n)$ with $|F_1| \le 2n-3$ and $|F_2| \le 2n-3$, but the vertex set pair (F_1, F_2) does not satisfy the condition in Theorem 5.2.2, i.e., there are no edges between $V(C\Gamma_n)\backslash(F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Suppose $V(C\Gamma_n) = F_1 \cup F_2$. By the definition of $C\Gamma_n$, $|F_1 \cup F_2| = |S_n| = n!$. It is obvious that $n! > 4n-6$ for $n \ge 4$. Since $n \ge 4$, we have that $n! = |V(C\Gamma_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \le |F_1| + |F_2| \le 2(2n-3) = 4n-6$, a contradiction. Therefore, $V(C\Gamma_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(C\Gamma_n) \setminus (F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$, and F_1 is a nature faulty set, $C\Gamma_n - F_1$ has two parts $C\Gamma_n - F_1 - F_2$ and $C\Gamma_n[F_2 \setminus F_1]$ (for convenience). Thus, $\delta(C\Gamma_n - F_1 - F_2) \ge 1$ and $\delta(C\Gamma_n[F_2 \setminus F_1]) \ge 1$. Similarly, $\delta(C\Gamma_n[F_1 \setminus F_2]) \ge 1$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a nature faulty set. Since there are no edges between $V(C\Gamma_n - F_1 - F_2)$ and $F_1 \bigtriangleup F_2$, $F_1 \cap F_2$ is a nature cut. Since $n \ge 4$, by Lemma 5.1.4, $|F_1 \cap F_2| \ge 2n-4$. By Lemma 5.2.4, $|F_2 \backslash F_1| \ge 2$. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \ge 2 + 2n - 4 = 2n-2$, which contradicts with that $|F_2| \le 2n-3$. So $C\Gamma_n$ is nature $(2n-3)$-diagnosable. By the definition of $t_1(C\Gamma_n)$, $t_1(C\Gamma_n) \ge 2n-3$. □

Combining Lemma 5.2.5 and 5.2.6, we have the following theorem.

Theorem 5.2.7 Let $n \ge 4$. Then the nature diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n under PMC model is $2n-3$.

5.3 The Nature Diagnosability of Cayley Graphs Generated by Transposition Trees under the MM* Model

Before discussing the nature diagnosability of the Cayley graph generated by the transposition tree under the MM* model, we firstly present the necessary and sufficient conditions of that a system (graph) G is g-good-neighbor t-diagnosable under MM* model.

Theorem 5.3.1 ([25, 112]) A system $G = (V,E)$ is g-good-neighbor t-diagnosable under the MM* model if and only if for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ satisfies one of the following conditions.

(1) There are two vertices $u,w \in V \setminus (F_1 \cup F_2)$ and there is a vertex $v \in F_1 \triangle F_2$ such that $uw \in E$ and $vw \in E$.

(2) There are two vertices $u,v \in F_1 \setminus F_2$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$.

(3) There are two vertices $u,v \in F_2 \setminus F_1$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$ (See Fig. 5.3).

Secondly, as we defined the nature faulty set and nature diagnosability and in chapter 2, it is straightforward to obtain the following theorem.

Theorem 5.3.2 A system $G = (V,E)$ is nature t-diagnosable under the MM* model if and only if for each distinct pair of nature faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ satisfies one of the following conditions.

(1) There are two vertices $u,w \in V \setminus (F_1 \cup F_2)$ and there is a vertex $v \in F_1 \triangle F_2$ such that $uw \in E$ and $vw \in E$.

(2) There are two vertices $u,v \in F_1 \setminus F_2$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$.

(3) There are two vertices $u,v \in F_2 \setminus F_1$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$ (See Fig. 5.3).

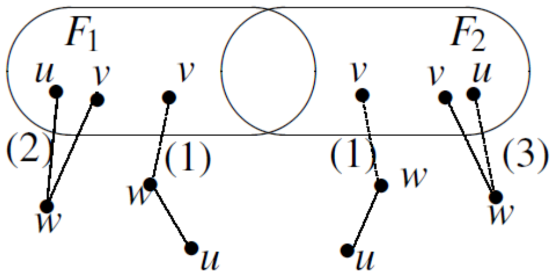

Fig. 5.3 Illustration of a distinguishable pair (F_1, F_2) under the MM* model.

Lemma 5.3.3 Let $n \geq 4$, the nature diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n under the MM* model $t_1(C\Gamma_n) \leq 2n-3$.

Proof: Let A, F_1 and F_2 be defined as in Lemma 5.2.3 (See Fig. 5.2). By Lemma 5.2.3, $|F_1| = 2n-4$, $|F_2| = 2n-2$, $\delta(C\Gamma_n - F_1) \geq 1$ and $\delta(C\Gamma_n - F_2) \geq 1$. So both F_1 and F_2 are nature faulty sets. By the definitions of F_1 and F_2, $F_1 \bigtriangleup F_2 = A$. Note $F_1 \setminus F_2 = \emptyset$, $F_2 \setminus F_1 = A$ and $(V(C\Gamma_n) \setminus (F_1 \cup F_2)) \cap A = \emptyset$. Therefore, both F_1 and F_2 do not satisfy any condition in Theorem 5.3.2, and $C\Gamma_n$ is not nature $(2n-2)$-diagnosable. Hence, $t_1(C\Gamma_n) \leq 2n-3$. The proof is completed. □

Lemma 5.3.4 Let $n \geq 4$. Then the nature diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n except the Bubble-sort graph B_4, under the MM* model $t_1(C\Gamma_n) \geq 2n-3$.

Proof: By the definition of nature diagnosability, it is sufficient to show that $C\Gamma_n$ is nature $(2n-3)$-diagnosable.

By Theorem 5.3.2, suppose, on the contrary, that there are two distinct nature faulty subsets F_1 and F_2 of $C\Gamma_n$ with $|F_1| \leq 2n-3$ and $|F_2| \leq 2n-3$, but the vertex set pair (F_1, F_2) does not satisfy the conditions in Theorem 5.3.2. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similar to the discussion on $V(C\Gamma_n) \neq F_1 \cup F_2$ in Lemma 5.2.6, we know that $V(C\Gamma_n) \neq F_1 \cup F_2$. Therefore, $V(C\Gamma_n) \neq F_1 \cup F_2$.

Claim I. $C\Gamma_n - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $C\Gamma_n - F_1 - F_2$ has at least one isolated vertex w, since F_1 is a nature faulty set, there is a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Since the vertex set pair (F_1, F_2) does not satisfy the conditions in Theorem 5.1, then there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Thus, there is just a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Similarly, we can show that there is just one vertex $v \in F_1 \setminus F_2$ such that v is adjacent to w when $F_1 \setminus F_2 \neq \emptyset$. Let $W \subseteq S_n \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $C\Gamma_n[S_n \setminus (F_1 \cup F_2)]$, and let H be the subgraph induced by the vertex set $S_n \setminus (F_1 \cup F_2 \cup W)$. Then for any $w \in W$, there are $(n-3)$ neighbors in $F_1 \cap F_2$ when $F_1 \setminus F_2 \neq \emptyset$. Since $|F_2| \leq 2n-3$, we have $\sum_{w \in W} |N_{C\Gamma_n[(F_1 \cap F_2) \cup W]}(w)| = |W|(n-3) \leq \sum_{v \in F_1 \cap F_2} d_{C\Gamma_n}(v) \leq |F_1 \cap F_2|(n-1) \leq (|F_2|-1)(n-1) \leq (2n-4)(n-1) = 2n^2-6n+4$. It follows that $|W| \leq 2n+4$. Note $|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq 2(2n-3)-(n-3) = 3n-3$. Suppose $V(H) = \emptyset$. Then $n! = |S_n| = |V(C\Gamma_n)| = |F_1 \cup F_2| + |W| \leq 3n-3+2n+4 = 5n+1$. This is a contradiction to the assumption that $n \geq 4$. So $V(H) \neq \emptyset$. Since the vertex set pair (F_1, F_2) does not satisfy the condition (1) of Theorem 5.1, and no vertex of $V(H)$ is isolated in H, we conclude that there is no edge between $V(H)$ and $F_1 \triangle F_2$. Thus, $F_1 \cap F_2$ is a vertex cut of $C\Gamma_n$ and $\delta(C\Gamma_n - (F_1 \cap F_2)) \geq 1$, i.e., $F_1 \cap F_2$ is a nature cut of $C\Gamma_n$. By Lemma 5.1.4, $|F_1 \cap F_2| \geq 2n-4$. Because $|F_1| \leq 2n-3$, $|F_2| \leq 2n-3$, and neither $F_1 \setminus F_2$ nor $F_2 \setminus F_1$ is empty, we have $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$. Let $F_1 \setminus F_2 = \{v_1\}$ and $F_2 \setminus F_1 = \{v_2\}$. Then for any vertex $w \in W$, w are adjacent to v_1 and v_2. According to Proposition 5.1.2, there are at most two common neighbors for any pair of vertices in $C\Gamma_n$, it follows that there are at most two isolated vertices in $C\Gamma_n - F_1 - F_2$.

Suppose that there is exactly one isolated vertex v in $C\Gamma_n - F_1 - F_2$ and $C\Gamma_n$ is a Star graph. Let v_1 and v_2 be adjacent to v. Then $N_{C\Gamma_n}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$. Since $C\Gamma_n$ contains no triangle, it follows that $N_{C\Gamma_n}(v_1) \setminus \{v\} \subseteq F_1 \cap F_2$; $N_{C\Gamma_n}(v_2) \setminus \{v\} \subseteq F_1 \cap F_2$; $[N_{C\Gamma_n}(v) \setminus \{v_1, v_2\}] \cap [N_{C\Gamma_n}(v_1) \setminus \{v\}] = \emptyset$ and $[N_{C\Gamma_n}(v) \setminus \{v_1, v_2\}] \cap [N_{C\Gamma_n}(v_2) \setminus \{v\}] = \emptyset$. Since $C\Gamma_n$ is a Star graph, by Proposition 5.1.1, there is at most one common neighbor for any pair of vertices in $C\Gamma_n$. Thus, it follows that $|[N_{C\Gamma_n}(v_1) \setminus \{v\}] \cap [N_{C\Gamma_n}(v_2) \setminus \{v\}]| = 0$. Thus, $|F_1 \cap F_2| \geq |N_{C\Gamma_n}(v) \setminus \{v_1, v_2\}| + |N_{C\Gamma_n}(v_1) \setminus \{v\}| + |N_{C\Gamma_n}(v_2) \setminus \{v\}| = (n-3)+(n-2)+(n-2)-0 = 3n-7$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1+3n-7 = 3n-6 > 2n-3$ $(n \geq 4)$, which contradicts $|F_2| \leq 2n-3$. Suppose $C\Gamma_n$ is not a Star graph. Then $C\Gamma_n$ contains a 4-cycle C_4. Let v_1 and v_2 be adjacent to v. Then $N_{C\Gamma_n}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$. Since

$C\Gamma_n$ contains no triangle, it follows that $N_{S_n}(v_1)\setminus\{v\}\subseteq F_1\cap F_2$, $N_{C\Gamma_n}(v_2)\setminus\{v\}\subseteq F_1\cap F_2$, $[N_{C\Gamma_n}(v)\setminus\{v_1,v_2\}]\cap[N_{S_n}(v_1)\setminus\{v\}]=\emptyset$ and $[N_{C\Gamma_n}(v)\setminus\{v_1,v_2\}]\cap[N_{C\Gamma_n}(v_2)\setminus\{v\}]=\emptyset$. Since there could be C_4 in $C\Gamma_n$, by Proposition 5.1.2, there is at most two common neighbors for any pair of vertices in $C\Gamma_n$. Thus, it follows that $|[N_{C\Gamma_n}(v_1)\setminus\{v\}]\cap[N_{C\Gamma_n}(v_2)\setminus\{v\}]|\leq 1$. Thus, $|F_1\cap F_2|\geq|N_{C\Gamma_n}(v)\setminus\{v_1,v_2\}|+|N_{C\Gamma_n}(v_1)\setminus\{v\}|+|N_{C\Gamma_n}(v_2)\setminus\{v\}|-|[N_{C\Gamma_n}(v_1)\setminus\{v\}]\cap[N_{C\Gamma_n}(v_2)\setminus\{v\}]|=(n-3)+(n-2)+(n-2)-1=3n-8$. Since $C\Gamma_n$ contains no B_4, it follows that $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\geq 1+3n-8=3n-7>2n-3\ \ (n\geq 5)$, which contradicts $|F_2|\leq 2n-3$.

Suppose that there are exactly two isolated vertices v and w in $C\Gamma_n-F_1-F_2$. Then $C\Gamma_n$ is not a Star graph. Combining this with $C\Gamma_4\neq B_4$, we obtain that $n\geq 5$. Let v_1 and v_2 be adjacent to v and w, respectively. Then $N_{C\Gamma_n}(v)\setminus\{v_1,v_2\}\subseteq F_1\cap F_2$. Since $C\Gamma_n$ contains no triangle, it follows that $N_{C\Gamma_n}(v_1)\setminus\{v,w\}\subseteq F_1\cap F_2$, $N_{C\Gamma_n}(v_2)\setminus\{v,w\}\subseteq F_1\cap F_2$, $[N_{C\Gamma_n}(v)\setminus\{v_1,v_2\}]\cap[N_{C\Gamma_n}(v_1)\setminus\{v,w\}]=\emptyset$ and $[N_{C\Gamma_n}(v)\setminus\{v_1,v_2\}]\cap[N_{C\Gamma_n}(v_2)\setminus\{v,w\}]=\emptyset$. Since $C\Gamma_n$ is not a Star graph, by Proposition 5.1.2 there are at most two common neighbors for any pair of vertices in $C\Gamma_n$. Thus, it follows that $|[N_{C\Gamma_n}(v_1)\setminus\{v,w\}]\cap[N_{C\Gamma_n}(v_2)\setminus\{v,w\}]|=0$. Thus, $|F_1\cap F_2|\geq|N_{C\Gamma_n}(v)\setminus\{v_1,v_2\}|+|N_{C\Gamma_n}(w)\setminus\{v_1,v_2\}|+|N_{C\Gamma_n}(v_1)\setminus\{v,w\}|+|N_{C\Gamma_n}(v_2)\setminus\{v,w\}|=(n-3)+(n-3)+(n-3)+(n-3)=4n-12$. It follows that $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\geq 1+4n-12=4n-11>2n-3\ \ (n\geq 5)$, which contradicts $|F_2|\leq 2n-3$.

Suppose $F_1\setminus F_2=\emptyset$. Then $F_1\subseteq F_2$. Since F_2 is a nature faulty set, $C\Gamma_n-F_2=S_n-F_1-F_2$ has no isolated vertex. The proof of Claim is completed.

Let $u\in V(C\Gamma_n)\setminus(F_1\cup F_2)$. By Claim I, u has at least one neighbor in $C\Gamma_n-F_1-F_2$. Since the vertex set pair (F_1,F_2) does not satisfy any condition in Theorem 5.3.2, by the condition (1) of Theorem 5.3.2, for any pair of adjacent vertices $u,w\in V(C\Gamma_n)\setminus(F_1\cup F_2)$, there is no vertex $v\in F_1\triangle F_2$ such that $uw\in E(C\Gamma_n)$ and $vw\in E(C\Gamma_n)$. It follows that u has no neighbor in $F_1\triangle F_2$. Since u is arbitrarily chosen, we know there is no edge between $V(C\Gamma_n)\setminus(F_1\cup F_2)$ and $F_1\triangle F_2$. Since $F_2\setminus F_1\neq\emptyset$ and F_1 is a nature faulty set, $\delta_{C\Gamma_n}([F_2\setminus F_1])\geq 1$. By Lemma 5.2.4, $|F_2\setminus F_1|\geq 2$. Since both F_1 and F_2 are nature faulty sets, and there is no edge between $V(C\Gamma_n)\setminus(F_1\cup F_2)$ and $F_1\triangle F_2$, $F_1\cap F_2$ is a nature cut of $C\Gamma_n$. By Lemma 5.1.4, we have $|F_1\cap F_2|\geq 2n-4$. Therefore, $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\geq 2+(2n-4)=2n-2$, which contradicts $|F_2| \leq 2n-3$. Therefore, $C\Gamma_n$ is nature $(2n-3)$-diagnosable and $t_1(C\Gamma_n) \geq 2n-3$. The proof is completed. $\square$

Combining Lemma 5.3.3 and 5.3.4, we have the following theorem.

Theorem 5.3.5 Let $n \geq 4$. Then the nature diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n except the Bubble-sort graph B_4 under MM* model is $2n-3$.

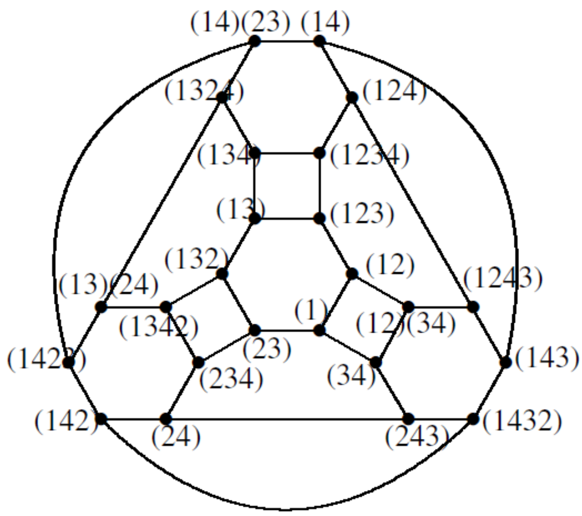

Fig. 5.4 The Bubble-sort graph B_4.

Next, we look at the Bubble-sort graph B_4 and have the following lemma.

Lemma 5.3.6 The nature diagnosability of the Bubble-sort graph B_4 (See Fig. 5.4) under the MM* model $t_1(B_4) \geq 4$.

Proof: By the definition of nature diagnosability, it is sufficient to show that B_4 is nature 4-diagnosable.

By Theorem 5.3.2, suppose, on the contrary, that there are two distinct nature faulty subsets F_1 and F_2 of B_4 with $|F_1| \leq 4$ and $|F_2| \leq 4$, but the vertex set pair (F_1, F_2) does not satisfy any condition in Theorem 5.3.2. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Note that $|V(B_4)| = 24$ and $|F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 8$. Therefore, $V(B_4) \neq F_1 \cup F_2$.

Similar to the Claim in Lemma 5.3.4, i.e., $C\Gamma_n - F_1 - F_2$ contains no isolated vertex, we know that $B_4 - F_1 - F_2$ has no isolated vertex.

Let $u \in V(B_4) \setminus (F_1 \cup F_2)$. Since $B_4 - F_1 - F_2$ has no isolated vertex, u has at least one neighbor in $B_4 - F_1 - F_2$. Since the vertex set pair (F_1, F_2) does not satisfy any condition in Theorem 5.3.2, by the condition (1) of Theorem 5.3.2, for any pair of adjacent vertices $u, w \in V(B_4) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(B_4)$ and $vw \in E(B_4)$. It follows that u has no neighbor in $F_1 \triangle F_2$. As u is chosen arbitrarily, there is no edge between $V(B_4) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a nature faulty set, $\delta_{B_4}([F_2 \setminus F_1]) \geq 1$. By Lemma 5.2.4, $|F_2 \setminus F_1| \geq 2$. Since both F_1 and F_2 are nature faulty sets, and there is no edge between $V(B_4) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a nature cut of B_4. By Lemma 5.1.4, we have $|F_1 \cap F_2| \geq 4$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 2 + 4 = 6$, which contradicts $|F_2| \leq 4$. Therefore, B_4 is nature 4-diagnosable and $t_1(C\Gamma_n) \geq 4$. The proof is completed. □

Finally, we point out that the nature diagnosability of the Bubble-sort graph B_4 under MM* model is not 5. In Fig. 5.1, let $F_1 = \{(23), (243), (1243), (123), (1)\}$ and $F_2 = \{(23), (243), (1243), (123), (12)(34))\}$. Then $B_4 - F_1 - F_2$ has two isolated vertices (12) and (34). It is easy to see that F_1 and F_2 are nature faulty subsets and $|F_1| = |F_2| = 5$ of B_4, but the vertex set pair (F_1, F_2) does not satisfy any condition in Theorem 5.3.2. By Lemma 5.3.3 and 5.3.6, we have the following proposition.

Proposition 5.3.1 The nature diagnosability of the Bubble-sort graph B_4 under the MM* model is 4.

5.4 Conclusion

In this chapter, we investigated the problem of nature diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n under the PMC model and MM* model. It is proved that nature diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n under the PMC model and MM* model is $2n-3$ except the Bubble-sort graph B_4 under MM* model, where $n \geq 4$, and the nature diagnosability of the Cayley graph B_4 under the MM* model is 4. The above results showed that the nature diagnosability is several times larger than the classical diagnosability of $C\Gamma_n$ based on the condition: nature.

Chapter 6

The 2-Good-Neighbor Diagnosability of Cayley Graphs Generated by Transposition Trees under the PMC Model & MM* Model

Let the Cayley graph $C\Gamma_n$ be generated by the transposition tree Γ_n. In this chapter, we study the 2-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model and MM* model and show that the diagnosability is $g(n-2)-1$, where $n \geq 4$ and g is the girth of $C\Gamma_n$. The results in this chapter is published in Theoretical Computer Science [84].

6.1 The 2-Good-Neighbor Diagnosability of Cayley Graphs Generated by Transposition Trees under the PMC Model

In this section, we will show that the 2-good-neighbor diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n under the PMC model.

Firstly, we include the 2-good-neighbor connectivity of Cayley graphs generated by transposition trees $C\Gamma_n$, which is a indispensable part in proof to determine the 2-good-neighbor diagnosability of $C\Gamma_n$ under PMC Model or MM*, where $n \geq 4$.

Theorem 6.1.1 ([108]) For $n \geq 4$, $\kappa^{(2)}(C\Gamma_n) = g(n-3)$, where g is the girth of $C\Gamma_n$.

By Theorem 6.1.1, we know that the 2-good-neighbor connectivity depends on the girth of of $C\Gamma_n$. Then we divide the Cayley graphs $C\Gamma_n$ generated by the transposition tree into two parts, one is the case if the girth of $C\Gamma_n$ is 4 and the other one is the case if the girth of $C\Gamma_n$ is 6.

In order to show the neighbourhood and the relevant properties of a 6-cycle in $C\Gamma_n$, we construct a 6-cycle as $A = \{(1),(12),(13),(23),(123),(132)\}$.

Theorem 6.1.2 Let A and $C\Gamma_n$ be defined as above, and let $F_1 = N_{C\Gamma_n}(A)$, $F_2 = A \cup N_{C\Gamma_n}(A)$.

(1) If $n \geq 4$ and the girth of $C\Gamma_n$ is 6, then $|F_1| = 6n-18$, $|F_2| = 6n-12$, $\delta(C\Gamma_n - F_1) \geq 2$, and $\delta(C\Gamma_n - F_2) \geq n-2 \geq 2$.

(2) If $n \geq 5$ and the girth of $C\Gamma_n$ is 4, then $|F_1| = 6n-18$, $|F_2| = 6n-12$, $\delta(C\Gamma_n - F_1) \geq 2$, and $\delta(C\Gamma_n - F_2) \geq n-3$.

Proof: Since $A = \{(1),(12),(13),(23),(123),(132)\}$, we have $C\Gamma_n[A] \cong C\Gamma_3$. Let $a \in A$, $a \neq (1)$, and let $4 \leq i \leq n$, $(xi) \in \Gamma_n$, then $a(xi) \in N_{C\Gamma_n}(A)$. Note $(1) \in A$ and $(yj) \in N_{C\Gamma_n}(A)$, where $4 \leq j \leq n$ and $(yj) \in \Gamma_n$. It is easy to see that $i \to i$ in the permutation a. Thus, $x \to i$ in the permutation $a(xi)$. Assume that $a(xi) = (yj)$, then $x = y$ or $x = j$. If $x = y$, then $i = j$. Thus, $(xi) = (yj)$, a contradiction to the fact that $a \neq (1)$. If $x = j$, then $i = y$. Thus, $(xi) = (yj)$, a contradiction to the fact that $a \neq (1)$. Therefore, $a(xi) \neq (yj)$. By Theorem 5.1.1, $C\Gamma_n$ is vertex transitive. Combining this with $a(xi) \neq (yj)$, we have that $|N_{C\Gamma_n}(u) \cap N_{C\Gamma_n}(v) \cap F_1| = 0$ for any $u, v \in A$. Thus from calculating, we have $|F_1| = 6(n-1-2) = 6n-18$, $|F_2| = |A| + |F_1| = 6n-12$.

In F_1, we find at most two vertices adjacent to one vertex x in $S_n \setminus F_2$. We consider two claims as following.

Claim 1. For any $x \in S_n \setminus F_2$, $|N_{C\Gamma_n}(x) \cap F_2)| \leq 1$ if the girth of $C\Gamma_n$ is 6.

In this case, $C\Gamma_n$ is a star. Let $\Omega = \{(12),(13),\ldots,(1n)\}$ and let $(1i),(1j),(1k),(1l) \in \Omega$, where $4 \leq i,j,k,l \leq n$. Note $(1i) \in F_1$. Assume $(1i)(1j) \in F_1$. Then there are $a \in A$ and $(1k) \in \Omega$ such that $(1i)(1j) = a(1k)$. Since the girth of the star is 6, we have $a \neq (1)$. Since $(1i)(1j) \in F_1$, $i \neq j$. It is easy to see that $1 \to j$ in the permutation $(1i)(1j)$ and $1 \to k$ in the permutation $a(1k)$. Since $(1i)(1j) = a(1k)$, $j = k$ and $(1i) = a$, a contradiction to $a \in A$. Therefore, $(1i)(1j) \in S_n \setminus F_2$ when $i \neq j$. Assume $(1i)(1j) = x = a(1k)(1l)$ when $i \neq j$ and $k \neq l$. It is easy to see that $1 \to j$ in the permutation $(1i)(1j)$ and $1 \to l$ in the permutation $a(1k)(1l)$. Since $(1i)(1j) = a(1k)(1l)$, $j = l$ and $(1i) = a(1k)$. Similarly, $i = k$ and $a = (1)$, a contradiction. Therefore, $(1i)(1j) \neq a(1k)(1l)$. By Theorem 5.1.1, It follows that $x \in V(C\Gamma_n - F_2)$ is at most adjacent to one vertex of F_2. Thus, for any $x \in S_n \setminus F_2$, $|N_{C\Gamma_n}(x) \cap F_2)| \leq 1$ if the girth of $C\Gamma_n$ is 6.

By Claim 1, $\delta(C\Gamma_n - F_2) \geq n-1-1 = n-2$. $C\Gamma_n - F_1$ has two components $C\Gamma_n - F_2$ and $C\Gamma_3$. Note that $\delta(C\Gamma_3) = 2$. Since $n \geq 4$, $\delta(C\Gamma_n - F_2) \geq n-2 \geq 2$, therefore, $\delta(C\Gamma_n - F_1) \geq 2$ for $n \geq 4$.

Claim 2. For any $x \in S_n \setminus F_2$, $|N_{C\Gamma_n}(x) \cap F_2)| \leq 2$ if the girth of $C\Gamma_n$ is 4.

In this case, $C\Gamma_n$ is not a star. Assume $(yi)(zj) = x = a(uk)(vl) \in S_n \setminus F_2$ and $a \neq (1)$. Note that $C\Gamma_3$ is a 6-cycle $v_1v_2v_3v_4v_5v_6v_1$. Let $v_1 = (1)$, since $C\Gamma_n$ is a bipartite graph, it has not add cycle. Therefore, $a = v_3$ or v_5. In this case, $v_1v_2v_3$ is in two 6-cycles $v_1v_2v_3v_4v_5v_6v_1$ and $v_1,v_2,v_3,a(uk),x,(yi),v_1$. This is a contradiction.

Assume $(yi)(zj) = x = a(uk)(vl) \in S_n \setminus F_2$ and $a = (1)$. Then there is a 4-cycle $(1),(yi),x,(uk),(1)$. Therefore, (yi) and (zj) are disjoint, and $(yi) = (vl)$, $(zj) = (uk)$. This 4-cycle is unique. Thus, for any $x \in S_n \setminus F_2$, $|N_{C\Gamma_n}(x) \cap F_2)| \leq 2$ if the girth of $C\Gamma_n$ is 4.

By Claim 2, $\delta(C\Gamma_n - F_2) \geq n-1-2 = n-3$. $C\Gamma_n - F_1$ has two components $C\Gamma_n - F_2$ and $C\Gamma_3$. Note that $\delta(C\Gamma_3) = 2$. Since $n \geq 5$, $\delta(C\Gamma_n - F_2) \geq n-3 \geq 2$. Therefore, $\delta(C\Gamma_n - F_1) \geq 2$ for $n \geq 5$. □

Similarly, in order to show the neighbourhood and the relevant properties of a 4-cycle in $C\Gamma_n$, we construct a 4-cycle as $A^* = \{(1),(12),(12)(34),(34)\}$, then $C\Gamma_4[A^*]$ is a 4-cycle, $N_{C\Gamma_4}(A^*) = \{(23),(123),(243),(1243)\}$. By Fig. 5.4, it is easy to see that $x \in S_4 \backslash (A^* \cup N_{C\Gamma_4}(A^*))$ is at most adjacent to one vertex of $N_{C\Gamma_4}(A^*)$.

Let (i,j) and (k,l) be disjoint, $(i,j),(k,l)\in\Gamma_n$, and let $A_1=\{(1),(i,j),(i,j)(k,l),(k,l)\}$,

$$A_1^*=\begin{cases} A^*, & n=4;\\ A_1, & n\geq 5.\end{cases}$$

Lemma 6.1.3 Let A_1^* be defined as above, and let the girth of $C\Gamma_n$ be 4. If $n\geq 4$, $F_1=N_{C\Gamma_n}(A_1^*)$ and $F_2=A_1^*\cup N_{C\Gamma_n}(A_1^*)$, then $|F_1|=4n-12$, $|F_2|=4n-8$, $\delta(C\Gamma_n-F_1)\geq 2$ and $\delta(C\Gamma_n-F_2)\geq 2$.

Proof: Suppose that $n=4$. Since the girth of $C\Gamma_4$ is 4, $C\Gamma_4$ is a Bubble-sort graph B_4. Let A^* be defined as above and let $F_1=N_{C\Gamma_4}(A^*)$ and $F_2=A^*\cup N_{C\Gamma_4}(A^*)$. Recall that $A^*=\{(1),(12),(12)(34),(34)\}$ and $C\Gamma_4[A^*]$ is a 4-cycle, it is straightforward that $N_{C\Gamma_4}(A^*)=\{(23),(123),(243),(1243)\}$ and so we have that $|F_1|=4=4*4-12$, $|F_2|=8=4*4-8$, $\delta(C\Gamma_4-F_1)\geq 2$ and $\delta(C\Gamma_4-F_2)\geq 2$. Therefore, suppose that $n\geq 5$. By $A_1=\{(1),(i,j),(i,j)(k,l),(k,l)\}$, we have that $C\Gamma_n[A_1]$ is a 4-cycle. Let $a\in A_1$ and $a\neq(1)$. If $(x,y)\in\Gamma_n$ and $(x,y)\notin A_1$, then $a(x,y)\in N_{C\Gamma_n}(A_1)$. Note $(1)\in A_1$ and $(rt)\in N_{C\Gamma_n}(A_1)$, where $(r,t)\in\Gamma_n$ and $(r,t)\notin A_1$. Assume that $a(x,y)=(r,t)$ and let $a=(i,j)$. If (i,j) and (x,y) are disjoint, then this is a contradiction to Theorem 5.1.2. Combining this with $(x,y)\notin A_1$, $|\{x,y\}\cap\{i,j\}|=1$. Without loss of generality, assume that $j=x$. Then $a(x,y)=(i,j,y)$, a contradiction to $a(x,y)=(r,t)$. Similarly, we have $a\neq(k,l)$. Let $a=(i,j)(k,l)$. If (x,y) is disjoint (i,j) or (k,l), then this is a contradiction to Theorem 5.1.2. Since $(x,y)\notin A_1$, we have that $|\{x,y\}\cap\{i,j\}|=1$ or $|\{x,y\}\cap\{k,l\}|=1$. It is easy to see that $(i,j)(k,l)(x,y)$ is not a transposition, a contradiction to $a(x,y)=(r,t)$. Therefore, $a(x,y)\neq(r,t)$. By Theorem 5.1.1, $C\Gamma_n$ is vertex transitive. Combining this with $a(x,y)\neq(r,t)$, we have that $|N_{C\Gamma_n}(u)\cap N_{C\Gamma_n}(v)\cap F_1|=0$ for any $u,v\in A_1$. By calculating, we have $|F_1|=4(n-1-2)=4n-12$, $|F_2|=|A_1|+|F_1|=4n-8$.

Claim. $|N_{C\Gamma_n}(x)\cap F_2)|\leq 2$ for any $x\in S_n\setminus F_2$.

In F_1 we find at most two vertices adjacent to one vertex x in $S_n\setminus F_2$. Let $(y,i),(z,j),(u,k)$, $(v,l)\in\Gamma_n$ and $a\in A_1$. Assume $(y,i)(z,j)=x=a(u,k)(v,l)\in S_n\setminus F_2$ and $a\neq(1)$. Note that $C\Gamma_n[A_1]$ is a 4-cycle $(1),(i,j),(i,j)(k,l),(k,l),(1)$. Since $C\Gamma_n$ is a bipartite graph, it has not add cycle. Therefore, $a = (i, j)(k,l)$. In this case, $(1),(i, j),(i, j)(k,l),a(uk),x,(yi),(1)$ is a 6-cycle. This is a contradiction. Thus, $a = (1)$.

Let $(x,y) \in \Gamma_n$ and $(x,y) \notin A_1$, $(r,t) \in \Gamma_n$, $(r,t) \notin A_1$. If (x,y) and (r,t) are disjoint, then $(1),(x,y),(x,y)(r,t),(r,t),(1)$ is a 4-cycle and $(x,y)(r,t) \in S_n \setminus F_2$. Note that this 4-cycle is unique. By Theorem 5.1.1, It follows that $x \in V(C\Gamma_n - F_2)$ is at most adjacent to two vertices of F_2. Thus, $|N_{C\Gamma_n}(x) \cap F_2)| \leq 2$ for any $x \in S_n \setminus F_2$. The proof of this claim is completed.

By Claim, $\delta(C\Gamma_n - F_2) \geq n-1-2 = n-3$. $C\Gamma_n - F_1$ has two components $C\Gamma_n - F_2$ and $C\Gamma_n[A_1]$. Note that $\delta(C\Gamma_n[A_1]) = 2$. Since $n \geq 5$, $\delta(C\Gamma_n - F_2) \geq n-3 \geq 2$. Therefore, $\delta(C\Gamma_n - F_1) \geq 2$ for $n \geq 5$. □

Next, we shall show the upper bounds of 2-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model, respectively.

Lemma 6.1.4 Let $H \subseteq V(C\Gamma_n)$ such that $\delta(C\Gamma_n[H]) \geq 2$. Then $|H| \geq g$, where g is the girth of $C\Gamma_n$.

The proof of the Lemma 6.1.4 is straightforward.

Lemma 6.1.5 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 6. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model $t_2(C\Gamma_n) \leq 6n-13$.

Proof: Let A be defined as above, and let $F_1 = N_{C\Gamma_n}(A)$, $F_2 = A \cup N_{C\Gamma_n}(A)$. By Theorem 6.1.2(1), $|F_1| = 6n-18$, $|F_2| = 6n-12$, $\delta(C\Gamma_n - F_1) \geq 2$ and $\delta(C\Gamma_n - F_2) \geq n-2 \geq 2$. Therefore, F_1 and F_2 are both 2-good-neighbor faulty sets of $C\Gamma_n$ with $|F_1| = 6n-18$ and $|F_2| = 6n-12$. Since $A = F_1 \triangle F_2$ and $N_{C\Gamma_n}(A) = F_1 \subset F_2$, there is no edge of $C\Gamma_n$ between $V(C\Gamma_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 5.2.1, we can deduce that $C\Gamma_n$ is not 2-good-neighbor $(6n-12)$-diagnosable under PMC model. Hence, by the definition of 2-good-neighbor diagnosability, we conclude that the 2-good-neighbor diagnosability of $C\Gamma_n$ is less than $6n-12$, i.e., $t_2(C\Gamma_n) \leq 6n-13$. □

Lemma 6.1.6 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 4. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model $t_2(C\Gamma_n) \leq 4n-9$.

Proof: Let A_1^* be defined as above, and let $F_1 = N_{C\Gamma_n}(A_1^*)$, $F_2 = A_1^* \cup N_{C\Gamma_n}(A_1^*)$. By Lemma 6.1.3, $|F_1| = 4n-12$, $|F_2| = 4n-8$, $\delta(C\Gamma_n - F_1) \geq 2$ and $\delta(C\Gamma_n - F_2) \geq 2$. Therefore, F_1 and F_2 are both 2-good-neighbor faulty sets of $C\Gamma_n$ with $|F_1| = 4n-12$ and $|F_2| = 4n-8$. Since $A_1 = F_1 \bigtriangleup F_2$ and $N_{C\Gamma_n}(A_1) = F_1 \subset F_2$, there is no edge of $C\Gamma_n$ between $V(C\Gamma_n)\backslash(F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$. By Theorem 5.2.1, we can deduce that $C\Gamma_n$ is not 2-good-neighbor $(4n-8)$-diagnosable under PMC model. Hence, by the definition of 2-good-neighbor diagnosability, we conclude that the 2-good-neighbor diagnosability of $C\Gamma_n$ is less than $4n-8$, i.e., $t_2(C\Gamma_n) \leq 4n-9$. □

Lemma 6.1.7 Let $n \geq 5$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 6. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model $t_2(C\Gamma_n) \geq 6n-13$.

Proof: By the definition of 2-good-neighbor diagnosability, it is sufficient to show that $C\Gamma_n$ is 2-good-neighbor $(6n-13)$-diagnosable. By Theorem 5.2.1, to prove $C\Gamma_n$ is 2-good-neighbor $(6n-13)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(C\Gamma_n)$ with $u \in V(C\Gamma_n)\backslash(F_1 \cup F_2)$ and $v \in F_1 \bigtriangleup F_2$ for each distinct pair of 2-good-neighbor faulty subsets F_1 and F_2 of $V(C\Gamma_n)$ with $|F_1| \leq 6n-13$ and $|F_2| \leq 6n-13$.

We prove this statement by contradiction. Suppose that there are two distinct 2-good-neighbor faulty subsets F_1 and F_2 of $V(C\Gamma_n)$ with $|F_1| \leq 6n-13$ and $|F_2| \leq 6n-13$, but the vertex set pair (F_1, F_2) does not satisfy the condition in Theorem 5.2.1, i.e., there are no edges between $V(C\Gamma_n)\backslash(F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \phi$. Assume $V(C\Gamma_n) = F_1 \cup F_2$. By the definition of $C\Gamma_n$, $|F_1 \cup F_2| = |S_n| = n!$. We claim that $n! > 12n-26$ for $n \geq 4$. When $n=4$, $n! = 24, 12n-26 = 22$. So $n! > 12n-26$ for $n=4$. Assume that $n! > 12n-26$ for $n \geq 5$. $(n+1)! = n!(n+1) > (n+1)(12n-26) = n(12n-26)+(12n-14)-12 = [12(n+1)-26]+n(12n-26)-12 = [12(n+1)-26]+2(6n^2-13n-6)$. It is sufficient to show that $6n^2-13n-6 \geq 0$ for $n \geq 5$.

Let $y = 6x^2 - 13x - 6$. Then $y = 6x^2 - 13x - 6$ is a quadratic function. If $x \geq 3$, we have $y = 6x^2 - 13x - 6 \geq 0$.

Since we have $n \geq 4$, then $n! = |V(C\Gamma_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(6n - 13) = 12n - 26$, a contradiction to $n! > 12n - 26$. Therefore, let $V(C\Gamma_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(C\Gamma_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a 2-good-neighbor faulty set, $C\Gamma_n - F_1$ has two components $C\Gamma_n - F_1 - F_2$ and $C\Gamma_n[F_2 \setminus F_1]$. Thus, $\delta(C\Gamma_n - F_1 - F_2) \geq 2$ and $\delta(C\Gamma_n[F_2 \setminus F_1]) \geq 2$. Similarly, $\delta(C\Gamma_n[F_1 \setminus F_2]) \geq 2$ when $F_1 \setminus F_2 \neq \phi$. Therefore, $F_1 \cap F_2$ is also a 2-good-neighbor faulty set. Since there are no edges between $V(C\Gamma_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 2-good-neighbor cut. Since $n \geq 5$, by Theorem 6.1.1, $|F_1 \cap F_2| \geq 6n - 18$. By Lemma 6.1.4, $|F_2 \backslash F_1| \geq 6$. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq 6 + 6n - 18 = 6n - 12$, which contradicts with that $|F_2| \leq 6n - 13$. So $C\Gamma_n$ is 2-good-neighbor $(6n - 13)$-diagnosable. By the definition of $t_2(C\Gamma_n)$, $t_2(C\Gamma_n) \geq 6n - 13$.

Based on these two cases above, we conclude that $t_2(C\Gamma_n) \geq 6n - 13$ if $g = 6$. □

In the end, we shall show the lower bounds of 2-good-neighbor diagnosability of $C\Gamma_n$ with girth 4 and 6 under the PMC model, respectively.

Lemma 6.1.8 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 4. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model $t_2(C\Gamma_n) \geq 4n - 9$.

Proof: By the definition of 2-good-neighbor diagnosability, it is sufficient to show that $C\Gamma_n$ is 2-good-neighbor $(4n - 9)$-diagnosable. By Theorem 5.2.1, to prove $C\Gamma_n$ is 2-good-neighbor $(4n - 9)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(C\Gamma_n)$ with $u \in V(C\Gamma_n) \backslash (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of 2-good-neighbor faulty subsets F_1 and F_2 of $V(C\Gamma_n)$ with $|F_1| \leq 4n - 9$ and $|F_2| \leq 4n - 9$.

We prove this statement by contradiction. Suppose that there are two distinct 2-good-neighbor faulty subsets F_1 and F_2 of $C\Gamma_n$ with $|F_1| \leq 4n - 9$ and $|F_2| \leq 4n - 9$, but the vertex set pair (F_1, F_2) does not satisfy the condition in Theorem 5.2.1, i.e., there are no edges between $V(C\Gamma_n) \backslash (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \phi$.

Assume $V(C\Gamma_n) = F_1 \cup F_2$. By the definition of $C\Gamma_n$, $|F_1 \cup F_2| = |S_n| = n!$. We claim that $n! > 8n-18$ for $n \geq 4$. When $n = 4$, $n! = 24, 8n-18 = 14$. So $n! > 8n-18$ for $n = 4$. Assume that $n! > 8n-18$ for $n \geq 5$. $(n+1)! = n!(n+1) > (n+1)(8n-18) = n(8n-18) + (8n-10) - 8 = [8(n+1)-18] + n(8n-18) - 8 = [8(n+1)-18] + 2(4n^2-9n-4)$. It is sufficient to show that $4n^2-9n-4 \geq 0$ for $n \geq 4$. Let $y = 4x^2-9x-4$. Then $y = 4x^2-9x-4$ is a quadratic function. If $x \geq 3$, then $y = 4x^2-9x-4 \geq 0$.

Since $n \geq 4$, we have that $n! = |V(C\Gamma_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(4n-9) = 8n-18$, a contradiction to $n! > 8n-18$. Therefore, let $V(C\Gamma_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(C\Gamma_n) \setminus (F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$, and F_1 is a 2-good-neighbor faulty set, $C\Gamma_n - F_1$ has two components $C\Gamma_n - F_1 - F_2$ and $C\Gamma_n[F_2 \setminus F_1]$. Thus, $\delta(C\Gamma_n - F_1 - F_2) \geq 2$ and $\delta(C\Gamma_n[F_2 \setminus F_1]) \geq 2$. Similarly, $\delta(C\Gamma_n[F_1 \setminus F_2]) \geq 2$ when $F_1 \setminus F_2 \neq \phi$. Therefore, $F_1 \cap F_2$ is also a 2-good-neighbor faulty set. Since there are no edges between $V(C\Gamma_n - F_1 - F_2)$ and $F_1 \bigtriangleup F_2$, $F_1 \cap F_2$ is a 2-good-neighbor cut. Since $n \geq 4$, by Theorem 6.1.1, $|F_1 \cap F_2| \geq g(n-3) = 4n-12$. By Lemma 6.1.4, $|F_2 \backslash F_1| \geq 4$. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq 4 + 4n - 12 = 4n-8$ for $g = 4$, which contradicts with that $|F_2| \leq 4n-9$. So $C\Gamma_n$ is 2-good-neighbor $(4n-9)$-diagnosable. By the definition of $t_2(C\Gamma_n)$, $t_2(C\Gamma_n) \geq 4n-9$.

Based on these two cases above, we conclude that $t_2(C\Gamma_n) \geq 4n-9$ if $g = 4$. □

Combining Lemma 6.1.5 and 6.1.7, we have the following theorem.

Theorem 6.1.9 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 6. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under PMC model is $6n-13$.

Combining Lemma 6.1.6 and 6.1.8, we have the following theorem.

Theorem 6.1.10 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 4. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model is $4n-9$.

6.2 The 2-Good-Neighbor Diagnosability of Cayley Graphs Generated by Transposition Trees under the MM* Model

Here we show the 2-good-neighbor diagnosability of $C\Gamma_n$ with girth 4 under the MM* model.

Lemma 6.2.1 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 6. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under the MM* model $t_2(C\Gamma_n) \leq 6n-13$.

Proof: Let A, F_1 and F_2 be defined in Theorem 6.1.2(1). By the Theorem 6.1.2(1), $|F_1| = 6n-18$, $|F_2| = 6n-12$, $\delta(C\Gamma_n - F_1) \geq 2$ and $\delta(C\Gamma_n - F_2) \geq n-2 \geq 2$. So both F_1 and F_2 are 2-good-neighbor faulty sets. By the definitions of F_1 and F_2, $F_1 \triangle F_2 = A$. Note $F_1 \setminus F_2 = \phi$, $F_2 \setminus F_1 = A$ and $(V(C\Gamma_n) \setminus (F_1 \cup F_2)) \cap A = \phi$. Therefore, both F_1 and F_2 does not satisfy any one condition in Theorem 5.3.1, and $C\Gamma_n$ is not 2-good-neighbor $(6n-12)$-diagnosable. Hence, $t_2(C\Gamma_n) \leq 6n-13$. The proof is completed. □

Lemma 6.2.2 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 4. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under the MM* model $t_2(C\Gamma_n) \leq 4n-9$.

Proof: Let A_1^*, F_1 and F_2 be defined in Lemma 6.1.3. By the Lemma 6.1.3, $|F_1| = 4n-12$, $|F_2| = 4n-8$, $\delta(C\Gamma_n - F_1) \geq 2$ and $\delta(C\Gamma_n - F_2) \geq 2$. So both F_1 and F_2 are 2-good-neighbor faulty sets. By the definitions of F_1 and F_2, $F_1 \triangle F_2 = A_1$. Note $F_1 \setminus F_2 = \phi$, $F_2 \setminus F_1 = A_1$ and $(V(C\Gamma_n) \setminus (F_1 \cup F_2)) \cap A_1 = \phi$. Therefore, both F_1 and F_2 does not satisfy any one condition in Theorem 5.3.1, and $C\Gamma_n$ is not 2-good-neighbor $(4n-8)$-diagnosable. Hence, $t_2(C\Gamma_n) \leq 4n-9$. The proof is completed. □

Lemma 6.2.3 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 6. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under the MM* model $t_2(C\Gamma_n) \geq 6n-13$.

Proof: By the definition of 2-good-neighbor diagnosability, it is sufficient to show that $C\Gamma_n$ is 2-good-neighbor $(6n-13)$-diagnosable. By Theorem 5.3.1, to prove $C\Gamma_n$ is

2-good-neighbor $(6n-13)$-diagnosable, it is equivalent to prove that for each distinct pair of 2-good-neighbor faulty subsets F_1 and F_2 of $V(C\Gamma_n)$ with $|F_1| \leq 6n-13$ and $|F_2| \leq 6n-13$ satisfies one of the following conditions.

(1). There are two vertices $u,w \in V(C\Gamma_n \setminus (F_1 \cup F_2)$ and there is a vertex $v \in F_1 \triangle F_2$ such that $uw \in E(C\Gamma_n$ and $vw \in E(C\Gamma_n)$.

(2). There are two vertices $u,v \in F_1 \setminus F_2$ and there is a vertex $w \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$ such that $uw \in E(C\Gamma_n)$ and $vw \in E(C\Gamma_n)$.

(3). There are two vertices $u,v \in F_2 \setminus F_1$ and there is a vertex $w \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$ such that $uw \in E(C\Gamma_n)$ and $vw \in E(C\Gamma_n)$.

Suppose, on the contrary, that there are two distinct 2-good-neighbor faulty subsets F_1 and F_2 of $C\Gamma_n$ with $|F_1| \leq 6n-13$ and $|F_2| \leq 6n-13$, but the vertex set pair (F_1,F_2) does not satisfy any condition in Theorem 5.3.1. Without loss of generality, assume that $F_2 \setminus F_1 \neq \phi$. Assume $V(C\Gamma_n) = F_1 \cup F_2$. By the definition of $C\Gamma_n$, $|F_1 \cup F_2| = |S_n| = n!$. We claim that $n! > 12n-26$ for $n \geq 4$. When $n=4$, $n! = 24, 12n-26 = 22$. So $n! > 12n-26$ for $n=4$. Assume that $n! > 12n-26$ for $n \geq 5$. $(n+1)! = n!(n+1) > (n+1)(12n-26) = n(12n-26) + (12n-14) - 12 = [12(n+1)-26] + n(12n-26) - 12 = [12(n+1)-26] + 2(6n^2-13n-6)$. It is sufficient to show that $6n^2-13n-6 \geq 0$ for $n \geq 5$. Let $y = 6x^2-13x-6$. Then $y = 6x^2-13x-6$ is a quadratic function. If $x \geq 3$, then $y = 6x^2-13x-6 \geq 0$.

Since $n \geq 4$, we have that $n! = |V(C\Gamma_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(6n-13) = 12n-26$, a contradiction to $n! > 12n-26$. Therefore, let $V(C\Gamma_n) \neq F_1 \cup F_2$.

Claim. $C\Gamma_n - F_1 - F_2$ has no isolated vertex.

Since F_1 is a 2-good-neighbor faulty set, $|N_{C\Gamma_n - F_1}(x)| \geq 2$ for any $x \in V(C\Gamma_n) \setminus F_1$. As the vertex set pair (F_1,F_2) does not satisfy any one condition in Theorem 5.3.1. By the condition (3) of Theorem 5.3.1, for any pair of vertices $u,v \in F_2 \setminus F_1$, there is no vertex $w \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$ such that $uw \in E(C\Gamma_n)$ and $vw \in E(C\Gamma_n)$. Therefore, any vertex w in $V(C\Gamma_n) \setminus (F_1 \cup F_2)$ has at most one neighbor in $F_2 \setminus F_1$. Thus, for any vertex $w \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$, $|N_{C\Gamma_n - F_1 - F_2}(w)| \geq 2-1 = 1$, i.e., every vertex of $C\Gamma_n - F_1 - F_2$ is not an isolated vertex. The proof of Claim is completed.

Let $u \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$. By Claim, u has at least one neighbor in $C\Gamma_n - F_1 - F_2$. Since the vertex set pair (F_1, F_2) does not satisfy any one condition in Theorem 5.3.1, by the condition (1) of Theorem 5.3.1, for any pair of adjacent vertices $u, w \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(C\Gamma_n)$ and $vw \in E(C\Gamma_n)$. It follows that u has no neighbor in $F_1 \triangle F_2$. By the arbitrariness of u, there is no edge between $V(C\Gamma_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a 2-good-neighbor faulty set, $\delta_{C\Gamma_n}([F_2 \setminus F_1]) \geq 2$. By Lemma 6.1.4, $|F_2 \setminus F_1| \geq 6$. Since both F_1 and F_2 are 2-good-neighbor faulty sets, and there is no edge between $V(C\Gamma_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 2-good-neighbor cut of $C\Gamma_n$. By Theorem 6.1.1, we have $|F_1 \cap F_2| \geq 6n - 18$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 6 + (6n - 18) = 6n - 12$, which contradicts $|F_2| \leq 6n - 13$. Therefore, $C\Gamma_n$ is 2-good-neighbor $(6n - 13)$-diagnosable and $t_2(C\Gamma_n) \geq 6n - 13$. The proof is completed. □

Lemma 6.2.4 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 4. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under the MM* model $t_2(C\Gamma_n) \geq 4n - 9$.

Proof: By the definition of 2-good-neighbor diagnosability, it is sufficient to show that $C\Gamma_n$ is 2-good-neighbor $(4n - 9)$-diagnosable. By Theorem 5.3.1, to prove $C\Gamma_n$ is 2-good-neighbor $(4n - 9)$-diagnosable, it is equivalent to prove that for each distinct pair of 2-good-neighbor faulty subsets F_1 and F_2 of $V(C\Gamma_n)$ with $|F_1| \leq 4n - 9$ and $|F_2| \leq 4n - 9$ satisfies one of the following conditions.

(1). There are two vertices $u, w \in V(C\Gamma_n \setminus (F_1 \cup F_2)$ and there is a vertex $v \in F_1 \triangle F_2$ such that $uw \in E(C\Gamma_n$ and $vw \in E(C\Gamma_n)$.

(2). There are two vertices $u, v \in F_1 \setminus F_2$ and there is a vertex $w \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$ such that $uw \in E(C\Gamma_n)$ and $vw \in E(C\Gamma_n)$.

(3). There are two vertices $u, v \in F_2 \setminus F_1$ and there is a vertex $w \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$ such that $uw \in E(C\Gamma_n)$ and $vw \in E(C\Gamma_n)$.

Suppose, on the contrary, that there are two distinct 2-good-neighbor faulty subsets F_1 and F_2 of $C\Gamma_n$ with $|F_1| \leq 4n - 9$ and $|F_2| \leq 4n - 9$, but the vertex set pair (F_1, F_2) does not satisfy any one condition in Theorem 5.3.1. Without loss of generality, assume that $F_2 \setminus F_1 \neq \phi$.

Assume $V(C\Gamma_n) = F_1 \cup F_2$. By the definition of $C\Gamma_n$, $|F_1 \cup F_2| = |S_n| = n!$. We claim that $n! > 8n-18$ for $n \geq 4$. When $n = 4$, $n! = 24, 8n-18 = 14$. So $n! > 8n-18$ for $n = 4$. Assume that $n! > 8n-18$ for $n \geq 5$. $(n+1)! = n!(n+1) > (n+1)(8n-18) = n(8n-18) + (8n-10) - 8 = [8(n+1)-18] + n(8n-18) - 8 = [8(n+1)-18] + 2(4n^2-9n-4)$. It is sufficient to show that $4n^2-9n-4 \geq 0$ for $n \geq 4$. Let $y = 4x^2-9x-4$. Then $y = 4x^2-9x-4$ is a quadratic function. If $x \geq 3$, then $y = 4x^2-9x-4 \geq 0$.

Since $n \geq 4$, we have that $n! = |V(C\Gamma_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(4n-9) = 8n-18$, a contradiction to $n! > 8n-18$. Therefore, let $V(C\Gamma_n) \neq F_1 \cup F_2$.

Claim. $C\Gamma_n - F_1 - F_2$ has no isolated vertex.

Since F_1 is a 2-good-neighbor faulty set, $|N_{C\Gamma_n - F_1}(x)| \geq 2$ for any $x \in V(C\Gamma_n) \setminus F_1$. As the vertex set pair (F_1, F_2) does not satisfy any one condition in Theorem 5.3.1. By the condition (3) of Theorem 5.3.1, for any pair of vertices $u, v \in F_2 \setminus F_1$, there is no vertex $w \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$ such that $uw \in E(C\Gamma_n)$ and $vw \in E(C\Gamma_n)$. Therefore, any vertex w in $V(C\Gamma_n) \setminus (F_1 \cup F_2)$ has at most one neighbor in $F_2 \setminus F_1$. Thus, for any vertex $w \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$, $|N_{C\Gamma_n - F_1 - F_2}(w)| \geq 2 - 1 = 1$, i.e., every vertex of $C\Gamma_n - F_1 - F_2$ is not an isolated vertex. The proof of Claim is completed.

Let $u \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$. By Claim, u has at least one neighbor in $C\Gamma_n - F_1 - F_2$. Since the vertex set pair (F_1, F_2) does not satisfy any condition in Theorem 5.3.1, by the condition (1) of Theorem 5.3.1, for any pair of adjacent vertices $u, w \in V(C\Gamma_n) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(C\Gamma_n)$ and $vw \in E(C\Gamma_n)$. It follows that u has no neighbor in $F_1 \triangle F_2$. By the arbitrariness of u, there is no edge between $V(C\Gamma_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a 2-good-neighbor faulty set, $\delta_{C\Gamma_n}([F_2 \setminus F_1]) \geq 2$. By Lemma 6.1.4, $|F_2 \setminus F_1| \geq 4$. Since both F_1 and F_2 are 2-good-neighbor faulty sets, and there is no edge between $V(C\Gamma_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 2-good-neighbor cut of $C\Gamma_n$. By Theorem 6.1.1, we have $|F_1 \cap F_2| \geq 4n-12$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 4 + (4n-12) = 4n-8$, which contradicts $|F_2| \leq 4n-9$. Therefore, $C\Gamma_n$ is 2-good-neighbor $(4n-9)$-diagnosable and $t_2(C\Gamma_n) \geq 4n-9$. The proof is completed. □

Combining Lemma 6.2.1 and 6.2.3, we have the following theorem.

Theorem 6.2.5 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 6. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under MM* model is $6n-13$.

Combining Lemma 6.2.2 and 6.2.4, we have the following theorem.

Theorem 6.2.6 Let $n \geq 4$, and let the girth of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n be 4. Then the 2-good-neighbor diagnosability of $C\Gamma_n$ under the MM* model is $4n-9$.

6.3 Conclusion

In this chapter, we investigated the problem of 2-good-neighbor diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n under the PMC model and MM* model. It is proved that 2-good-neighbor diagnosability of the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n under the PMC model and MM* model is $g(n-2)-1$, where $n \geq 4$ and g is the girth of $C\Gamma_n$. The above results showed that the 2-good-neighbor diagnosability is several times larger than the classical diagnosability of $C\Gamma_n$ depending on the condition 2-good-neighbor. Comparing with 1-good-neighbor property, 2-good-neighbor property requires that the minimum degree of each component is 2 after the removal of faulty set. Thus, it is easy to see there exists a cycle in each component. It is more complicated to investigate the neighborhood of minimum cycles. Therefore, the results are different for Cayley graphs with different girths.

Chapter 7

The Nature Connectivity and Diagnosability of Cayley Graphs Generated by Complete Graphs

In this chapter, we show that the connectivity of CK_n is $\frac{n(n-1)}{2}$, the nature connectivity of CK_n is n^2-n-2 and the nature diagnosability of CK_n under the PMC model is n^2-n-1 for $n \geq 4$ and under the MM* model is n^2-n-1 for $n \geq 5$. The results in this chapter is published in Discrete Applied Mathematics [87].

7.1 Background & Known Results

Firstly, we list a few known results in order to prove Proposition 7.1.2, which will play an important role of determining the nature connectivity and diagnosabilities of CK_n.

Theorem 7.1.1 ([1]) The nest graph CK_n is vertex transitive and bipartite.

Proposition 7.1.1 Let $n \geq 3$. The girth of CK_n is 4.

Proof: Since the nest graph is a simple graph, it is easy to see that the girth is not 2. By Theorem 7.1.1, there is no 3-cycle in CK_n. Note that there is a 4-cycle in CK_n as,

$(1), (ab), (ab)(cd), (cd), (1)$, where (ab) is disjoint to (cd). Therefore, the girth of CK_n is 4.
□

By using Theorem 7.1.1 and Proposition 7.1.1 in the proofs, we will have the following proposition and lemma.

Proposition 7.1.2 Let CK_n be a nest graph. If two vertices u, v are adjacent, then there is no common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| = 0$. If two vertices u, v are not adjacent, then there are at most three common neighbor vertices of these two vertices, i.e., $|N(u) \cap N(v)| \leq 3$.

Proof: In this proof, a permutation is denoted by a product of disjoint cycles. The two cases can be proved by contradiction.

Case 1. If two vertices are adjacent and they have a common neighbor vertex, then these 3 vertices will form a cycle of length 3. It is a contradiction to Theorem 7.1.1 that there are no odd cycles in a bipartite graph CK_n.

Case 2. Let two vertices be non-adjacent. Suppose, on the contrary, that $|N(u) \cap N(v)| \geq 4$. By Theorem 7.1.1, without loss of generality, assume that $u = (1)$, i.e., u is the identity vertex. Then $v \notin E(K_n)$. It is sufficient to suppose that $\{(ia), (jb), (kc), (ld)\} \subseteq E(K_n)$, $\{(ia), (jb), (kc), (ld)\} \subseteq N(u) \cap N(v)$ and $|\{(ia), (jb), (kc), (ld)\}| = 4$. By Proposition 7.1.1, the girth of CK_n is 4. Let $v = (ia)(jb)$.

Case 2.1. (ia) is disjoint to (jb).

In this case, $u, (ia), v, (jb), u$ is also a 4-cycle. Since $u, (ia), v, (kc), u$ is also a cycle of length 4, let $v = (kc)(xy) = (ia)(jb)$. By Theorem 5.1.2, we have that (kc) is disjoint to (xy), and $(kc) = (jb)$ or $(kc) = (ia)$, a contradiction. Similarly, we have $(ld) = (jb)$ or $(ld) = (ia)$, a contradiction. Therefore, $|N(u) \cap N(v)| = 2$ in this case.

Case 2.2. (ia) is disjoint to (jb).

Without loss of generality, let $a = j$. We have $v = (iab)$. Since $u, (ia), v, (jb), u$ is a 4-cycle, there is $(xy) \in E(K_n)$ such that $(jb)(xy) = (ab)(xy) = v = (iab)$. By Theorem 5.1.2, one of $\{x, y\}$ is i. Let $i = x$. Then $y = a$ or $y = b$. When $y = a$, $(ab)(xy) = (ab)(ia) = (iba)$, a contradiction. When $y = b$, $(ab)(xy) = (ab)(ib) = (iab)$. Let $i = y$. Then $x = a$ or

$x = b$. When $x = a$, $(ab)(xy) = (ab)(ai) = (iba)$, a contradiction. When $x = b$, $(ab)(xy) = (ab)(bi) = (ab)(ib) = (iab)$. Therefore, (xy) can only be (ib). Similarly, we can discuss other situations. Therefore, (iab) could only be decomposed as follows, $v = (iab) = (ia)(ab) = (ab)(ib) = (ib)(ia)$. We have $\{(ia),(jb),(kc),(ld)\} = \{(ia),(ab),(ib)\}$, which is a contradiction to $|\{(ia),(jb),(kc),(ld)\}| = 4$. □

Lemma 7.1.2 There are $(n-1)!$ independent cross-edges between two different H_i's in CK_n and a vertex of $V(H_i)$ is adjacent to exactly one vertex of $V(H_j)$ for $i, j \in \{1,2,\ldots,n\}$.

Proof: We prove by contradiction. Let $Cay(H,S_n)$ be decomposed as in the last position. Without loss of generality, we discuss the situation between H_1 and H_2. Then the last position of vertex in H_1 is i while it is j in H_2, where $i, j \in \{1,\ldots,n\}$. Since there is no 3-cycle in CK_n, we suppose v_1 and v_2 are two nonadjacent vertices in H_1 and are adjacent to a common vertex v_3 in H_2. Note that v_3 is a transposition from the generating set, which includes n, to be adjacent to the vertex in H_1. Let it be (r,n). Since the n-th position of vertex in H_1 is i, we have i is on the r-th position in v_3. Then we have $v_3(r,n) = v_1 = v_2$, a contradiction. Therefore, v_3 is only adjacent to v_1 in $V(H_1)$. By the arbitrariness of v_1 and v_3, we have that a vertex of $V(H_i)$ is adjacent to exactly one vertex of $V(H_j)$ for $i, j \in \{1,2,\ldots,n\}$. Combining this with that there are $(n-1)!$ vertices in H_i, we have that there are $(n-1)!$ independent cross-edges between two different H_i's in CK_n. □

7.2 The Connectivity of Cayley Graphs Generated by Complete Graphs

In this section we will examine the connectivity of CK_n, which will help us determine the nature connectivity and diagnosabilities of CK_n.

Theorem 7.2.1 For $n \geq 3$, the connectivity of CK_n is $\frac{n(n-1)}{2}$, i.e., $\kappa(CK_n) = \frac{n(n-1)}{2}$.

Proof: We prove it by induction on n. When $n = 3$, it is easy to see that $\kappa(CK_3) = \frac{n(n-1)}{2} = 3$ since CK_3 is isomorphic to $K_{3,3}$. We decompose CK_n along the last position, denoted by H_i $(i = 1,\ldots,n)$. Then H_i and CK_{n-1} are isomorphic. Let F be the faulty vertex set in CK_n, $F \le \frac{n(n-1)}{2} - 1$ and $F_i = H_i \cap F$. When $n = 4$, without loss of generality, let $|F_1| \ge |F_2| \ge |F_3| \ge |F_4|$. If $3 \le |F_1| \le 5$, we have $F_4 = \emptyset$. According to the Lemma 7.1.2, each vertex in H_i is adjacent to a vertex in $H_4 = H_4 - F_4$, where $i \in \{1,2,3\}$. Then $CK_4 - F$ is connected. Assume $|F_i| \le 2$. Combining this with that H_i is isomorphic with CK_3, $H_i - F_i$ is connected. Since $|E_{i,j}(CK_4)| = (n-1)! = 6 > 4 \ge |F_i| + |F_j|$, we have $CK_4[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected, where $i, j \in \{1,2,3,4\}$. Therefore, $CK_4 - F$ is connected and $\kappa(CK_4) \ge 6$. Since $\delta(CK_4) = 6 \ge \kappa(CK_4) \ge 6$, we have $\kappa(CK_4) = 6$. When $n = k-1$, we assume that $\kappa(CK_{k-1}) = \delta(CK_{k-1}) = \frac{(k-1)(k-2)}{2}$. When $n = k$, let $F \le \frac{k(k-1)}{2} - 1$ and $|F_1| \ge |F_2| \ge \ldots \ge |F_k|$. If $\frac{(k-1)(k-2)}{2} \le |F_1| \le \frac{k(k-1)}{2} - 1$, we have $\sum_{i=2}^{k} |F_i| \le (k-2)$, then $F_k = \emptyset$. According to the Lemma 7.1.2, each vertex in H_i is adjacent to a vertex in $H_k = H_k - F_k$, where $i \in \{1,\ldots,k-1\}$. Then $CK_k - F$ is connected. If $|F_i| \le \frac{(k-1)(k-2)}{2} - 1$, by assumption, $H_i - F_i$ is connected. Since $|E_{i,j}(CK_k)| = (k-1)! > k^2 - 3k \ge |F_i| + |F_j|$ for $k \ge 4$, we have we have $CK_k[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected, where $i, j \in \{1,2,\ldots,n\}$. Therefore, CK_k is connected and $\kappa(CK_k) \ge \frac{k(k-1)}{2}$. Since $\delta(CK_k) = \frac{k(k-1)}{2} \ge \kappa(CK_k) \ge \frac{k(k-1)}{2}$, we have $\kappa(CK_k) = \frac{k(k-1)}{2}$. Therefore, $\kappa(CK_n) = \frac{n(n-1)}{2}$. □

7.3 The Nature Connectivity of Cayley Graphs Generated by Complete Graphs

In this section we will determine the nature connectivity of CK_n, which will help us to study the diagnosabilities of CK_n under PMC model and MM*.

Lemma 7.3.1 The nature connectivity of the Cayley graph CK_4 generated by the complete graph K_4 is not smaller than 10, i.e., $\kappa^*(CK_4) \ge 10$.

Proof: We decompose CK_4 along the last position, denoted by H_i $(i = 1,2,3,4)$. Then H_i and CK_3 are isomorphic. The edges whose end vertices are in different H_i's are called the cross-edges with respect to the given decomposition. Note that each vertex is incident to $(n-1) = 3$ cross-edges and there are $(n-1)! = 6$ independent cross-edges between two different H_i's by Lemma 7.1.2. Let F be a nature cut of CK_4 such that $|F| \leq 9$ and $F_i = F \cap V(H_i)$. Without loss of generality, let $|F_1| \geq |F_2| \geq |F_3| \geq |F_4|$.

Case 1. $|F_4| = 0$.

Since each of $V(H_i)$ for $i \in \{1,2,3\}$ is adjacent to one vertex in $H_4 - F_4 = H_4$, we have that $CK_4 - F$ is connected, a contradiction to that F is a nature cut of CK_4.

Case 2. $|F_4| = 1$.

By Theorem 7.2.1, we have $H_4 - F_4$ is connected. Let $F_4 = \{u\}$. Note that there is only one vertex u_i in H_i for $\{1,2,3\}$ such that u_i is adjacent to u. If $u_i \in F$ for $i \in \{1,2,3\}$, we have that $CK_4[V(H_i - F_i) \cup V(H_4 - F_4)]$ is connected. Thus, $CK_4 - F$ is connected, a contradiction to that F is a nature cut of CK_4. Then there is at least one $u_i \notin F$, without loss of generality, let it be u_1. Since F is a nature cut of CK_4, we know that $CK_4 - F$ has no isolated vertex and hence $d_{CK_4 - F}(u_1) \geq 1$. Combining this with the fact that u_1 is only adjacent to u in H_4, we have that there is a vertex u_1' in $CK_4 - (F \cup V(H_4))$ such that u_1' is adjacent to u_1. Moreover, there is no 3-cycle in CK_4, which implies that u_1' is not adjacent to u. Therefore, u_1' is adjacent to one vertex in $H_4 - F_4$ and $CK_4[V(H_4 - F_4) \cup \{u_1', u_1\}]$ is connected. For other vertices in $H_1 - F_1$, each of them is adjacent to exactly one vertex in $H_4 - F_4$. Then we have $CK_4[V(H_4 - F_4) \cup V(H_1 - F_1) \cup \{u_1'\}]$ is connected. The cases of $H_2 - F_2$ and $H_3 - F_3$ are similar. From the above, we have $CK_4 - F$ is connected, a contradiction to that F is a nature cut of CK_4.

Case 3. $|F_4| = 2$.

For $|F| \leq 5$, by Theorem 7.2.1, we have that $CK_4 - F$ is connected, a contradiction to that F is a nature cut of CK_4. Therefore, let $6 \leq |F| \leq 9$. Combining this with that $|F_1| \geq |F_2| \geq |F_3| \geq |F_4|$, we have $|F_2| = |F_3| = |F_4| = 2$ and $2 \leq |F_1| \leq 3$. Suppose $|F_1| = 2$. Note that H_i is isomorphic to CK_3. By Theorem 7.2.1, $CK_3 - F_i$ is connected. Therefore, we have $H_i - F_i$ is connected. Since each of $V(H_i)$ is adjacent to one vertex in H_j for $i, j \in \{1,2,3,4\}$ and $|E_{i,j}(CK_4)| = (n-1)! = 6 > 4 \geq |F_i| + |F_j|$, we have $CK_4[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected. Therefore, $CK_4 - F$ is connected, a contradiction to that F is a nature cut of CK_4. Then suppose $|F_1| = 3$. Assume that there is no isolated vertices in $H_1 - F_1$. Since $|E_{1,i}(CK_4)| = (n-1)! = 6 > 5 = |F_1| + |F_i|$, we have that $CK_4[V(H_1 - F_1) \cup V(H_i - F_i)]$ is connected. Therefore, we have $CK_4 - F$ is connected, a contradiction to that F is a nature cut of CK_4. Then there are isolated vertices in $H_1 - F_1$. Since $H_1 = K_{3,3}$, $H_1 - F_1$ has three isolated vertices. Since there are no isolated vertex in $CK_4 - F$, these isolated vertices in $H_1 - F_1$ are adjacent to vertices in $CK_4 - F - H_1$, respectively. Since $|E_{i,j}(CK_4)| = (n-1)! = 6 > 4 \geq |F_i| + |F_j|$ for $i, j \in \{2,3,4\}$, we have $CK_4[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected. Therefore, $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_4.

Therefore, F is not a nature cut of CK_4 when $|F| \leq 9$ and $\kappa^*(CK_4) \geq 10$. □

Lemma 7.3.2 The nature-connectivity of the Cayley graph CK_5 generated by the complete graph K_5 is not smaller than 18, i.e., $\kappa^*(CK_5) \geq 18$.

Proof: We decompose CK_5 as in the last position, denoted by H_i $(i = 1,2,3,4,5)$. Then H_i and CK_4 are isomorphic. The edges whose end vertices are in different H_i's are called the cross-edges with respect to the given decomposition. Note that each vertex is incident to $(n-1) = 4$ cross-edges and there are $(n-1)! = 24$ independent cross-edges between two different H_i's by Lemma 7.1.2. Let F be a nature cut of CK_5 such that $|F| \leq 17$ and $F_i = F \cap V(H_i)$. Without loss of generality, let $|F_1| \geq |F_2| \geq |F_3| \geq |F_4| \geq |F_5|$.

Case 1. $|F_5| = 0$.

Since each of $V(H_i)$ for $i \in \{1,2,3,4\}$ is adjacent to one vertex in $H_5 - F_5 = H_5$, we have that $CK_5 - F$ is connected, a contradiction to that F is a nature cut of CK_5.

Case 2. $|F_5| = 1$.

By Theorem 7.2.1, we have $H_5 - F_5$ is connected. Let $F_5 = \{u\}$. Note that there is only one vertex u_i in H_i for $\{1,2,3,4\}$ such that u_i is adjacent to u. If $u_i \in F$ for $i \in \{1,2,3,4\}$, we have that $CK_5[V(H_i - F_i) \cup V(H_5 - F_5)]$ is connected. Thus, $CK_5 - F$ is connected, a contradiction to that F is a nature cut of CK_5. Then there is at least one $u_i \notin F$, without loss of generality, let it be u_1. Since F is a nature cut of CK_5, we have that $CK_5 - F$ has no isolated vertex and hence $d_{CK_5 - F}(u_1) \geq 1$. Combining this with that u_1 is only adjacent to u in H_5, we have there is a vertex u_1' in $CK_5 - (F \cup V(H_5))$ such that u_1' adjacent to u_1. Moreover, there is no 3-cycle in CK_5, which implies that u_1' is not adjacent to u. Therefore, u_1' is adjacent to one vertex in $H_5 - F_5$ and $CK_5[V(H_5 - F_5) \cup \{u_1', u_1\}]$ is connected. For other vertices in $H_1 - F_1$, each of them is adjacent to exactly one vertex in $H_5 - F_5$. Then we have $CK_5[V(H_5 - F_5) \cup V(H_1 - F_1) \cup \{u'_1\}]$ is connected. The case of $H_i - F_i$ for $i \in \{2,3,4\}$ is similar. From the above, we have $CK_5 - F$ is connected, a contradiction to that F is nature cut of CK_5.

Case 3. $|F_5| = 2$.

It is easy to see that $|F_1| + |F_2| \leq 17 - 3 \times 2 = 11$, which implies only F_1 could have $|F_1| \geq 6 = \kappa(CK_4)$ by Theorem 7.2.1. On the other hand, we have $|F_1| \leq 17 - 4 \times 2 = 9 < 10 \leq \kappa^*(CK_4)$ by Lemma 7.3.1. Suppose $6 \leq |F_1| \leq 9$. Note that $|F_i| < 6$ for $i \in \{2,3,4,5\}$. We have that $H_i - F_i$ for $i \in \{2,3,4,5\}$ is connected by Theorem 7.2.1. Since $|E_{i,j}(CK_5)| = (n-1)! = 24 > 7 \geq |F_i| + |F_j|$, we have $CK_5[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected, where $i, j \in \{2,3,4,5\}$. Suppose that $H_1 - F_1$ has no isolated vertices. Since $|E_{1,i}(CK_5)| = (n-1)! = 24 > 11 \geq |F_1| + |F_i|$, we have $CK_5[V(H_1 - F_1) \cup V(H_i - F_i)]$ is connected. Therefore, we have $CK_5 - F$ is connected, a contradiction to that F is nature cut of CK_5. Then there are isolated vertices in $H_1 - F_1$. Since there is no isolated vertex in $CK_5 - F$, these isolated vertices in $H_1 - F_1$ are adjacent to vertices in $CK_5 - F - V(H_1)$, respectively. On the other hand, we suppose that there is a component G_1 in $H_1 - F_1$ such that $|V(G_1)| = 2$. Since $|N_{H_1}(V(G_1))| = (n-1)(n-2) - 2 = 10 > 9 \geq |F_1|$, a contradiction. Therefore, we have $|V(G_1)| \geq 3$. Since $3(n-1) = 12 > 17 - 6 = 11$, we have $CK_n[V(G_1) \cup V(H_i)]$ is connected for at least one $i \in \{2,3,4,5\}$. The cases of other components in $H_1 - F_1$ are similar. From the above, $CK_5 - F$ is connected, a contradiction to that F is nature cut of CK_5.

Case 4. $|F_5| = 3$.

It is easy to see that $|F_1| \leq 17 - 4 \times 3 = 5$. Then we have $|F_i| \leq \kappa(CK_4) = \frac{n(n-1)}{2} = 6$ for $i \in \{1,2,3,4,5\}$ by Theorem 7.2.1. Thus, $H_i - F_i$ is connected. On the other hand, $|E_{i,j}(CK_5)| = (n-1)! = 24 > 8 \geq |F_i| + |F_j|$, we have $CK_5[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected. Therefore, $CK_5 - F$ is connected, a contradiction to that F is nature cut of CK_5.

Therefore, F is not a nature cut of CK_5 when $|F| \leq 17$ and $\kappa^*(CK_5) \geq 18$. $\square$

Theorem 7.3.3 For $n \geq 4$, the nature-connectivity of the Cayley graph CK_n generated by the complete graph K_n is $n^2 - n - 2$, i.e., $\kappa^*(CK_n) = n^2 - n - 2$.

Proof: By Proposition 7.1.1, the girth of CK_n is 4. By Theorem 2.4.3, let $(12) \in E(K_n)$ and $A = \{(1), (12)\}$. Then $CK_n[A] = K_2$. Since CK_n has no 3-cycles and its regularity is $\frac{n(n-1)}{2}$, we have $|N_{CK_n}(A)| = n(n-1) - 2 = n^2 - n - 2$. Let $F_1 = N_{CK_n}(A)$ and $F_2 = A \cup N_{CK_n}(A)$.

Note that $|N((1)) \setminus (12)| = \frac{n(n-1)}{2} - 1$. Let $a \in (N((1)) \setminus (12))$. For any $x \in S_n \setminus F_2$, suppose that a is adjacent to x. Since CK_n is bipartite, x is not adjacent to any vertex of $(N((12)) \setminus (1))$. Therefore, $d_{CK_n[S_n \setminus F_2]}(x) \geq \frac{n(n-1)}{2} - (\frac{n(n-1)}{2} - 1) = 1$ and $\delta(CK_n - F_1 - F_2) \geq 1$. Then F_1 is a nature cut of CK_n. Therefore, $K^{(1)}(CK_n) \leq n^2 - n - 2$. It is sufficient to show that F is not a nature cut of CK_n when $|F| \leq n^2 - n - 3$.

We decompose CK_n along the last position, denoted by H_i $(i = 1, 2, \ldots, n)$. Then H_i and CK_{n-1} are isomorphic. The edges whose end vertices are in different H_i's are called the cross-edges with respect to the given decomposition. Note that each vertex is incident to $(n-1)$ cross-edges and there are $(n-1)!$ independent cross-edges between two different H_i's by Lemma 7.1.2. Let $F_i = F \cap V(H_i)$. Without loss of generality, let $|F_1| \geq |F_2| \geq \ldots \geq |F_{n-1}| \geq |F_n|$. This claim is shown by induction on n. For $4 \leq n \leq 5$, by Lemmas 7.3.1 and 7.3.2, F is not a nature cut of CK_n when $|F| \leq n^2 - n - 3$. Assume that F is not a nature cut of CK_{n-1} when $|F| \leq (n-1)^2 - (n-1) - 3$. Now consider CK_n when $n \geq 6$. Suppose, on the contrary, that F is a nature cut of CK_n when $|F| \leq n^2 - n - 3$. We discuss the following cases.

Case 1. $|F_1| < \frac{(n-1)(n-2)}{2}$.

Since $|F_1| \geq |F_2| \geq \ldots \geq |F_{n-1}| \geq |F_n|$ and $|F_1| < \frac{(n-1)(n-2)}{2}$, we have $|F_i| < \frac{(n-1)(n-2)}{2}$ for every i. By Theorem 7.2.1, $H_i - F_i$ is connected. Since there are $(n-1)!$ independent cross-edges between two H_i's and $(n-1)! \geq 2 \cdot \frac{(n-1)(n-2)}{2} > |F_i| + |F_j|$, we have $CK_n[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected. Therefore, $CK_n - F$ is connected, a contradiction to that F is nature cut of CK_n.

Case 2. $\frac{(n-1)(n-2)}{2} \leq |F_1| \leq (n-1)^2 - (n-1) - 3$.

Since $2 \cdot \frac{(n-1)(n-2)}{2} < n^2 - n - 3 < 3 \cdot \frac{(n-1)(n-2)}{2}$ for $n \geq 6$, only F_2 could have that $\frac{(n-1)(n-2)}{2} \leq |F_2| \leq (n-1)^2 - (n-1) - 3$, other F_i for $i \in \{3, \ldots, n\}$ has that $|F_i| < \frac{(n-1)(n-2)}{2}$.

Case 2.1. $\frac{(n-1)(n-2)}{2} \leq |F_2| \leq (n-1)^2 - (n-1) - 3$.

Since $\frac{(n-1)(n-2)}{2} \leq |F_i| \leq (n-1)^2-(n-1)-3$ for $i \in \{1,2\}$, by the inductive hypothesis, $H_i - F_i$ either has isolated vertices or is connected.

Case 2.1.1. Neither $H_1 - F_1$ nor $H_2 - F_2$ has isolated vertices.

In this case, each of $\{H_1 - F_1, H_2 - F_2\}$ is connected. Since $|F \setminus (F_1 \cup F_2)| < \frac{(n-1)(n-2)}{2}$, similarly to Case 1, $CK_n[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected, where $i, j \in \{3, \ldots, n\}$. Since $|V(H_1 - F_1)|(n-2) \geq [(n-1)! - (n-1)^2 + (n-1) + 3](n-2) > (n^2-n-3) - (n-1)(n-2) \geq |F \setminus (F_1 \cup F_2)|$, we have $CK_n[V(H_1 - F_1) \cup V(H_i - F_i)]$ is connected for at least one $i \in \{3, \ldots, n\}$. The case of $H_2 - F_2$ is similar. Therefore, $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_n.

Case 2.1.2. One of $H_1 - F_1$ and $H_2 - F_2$ has isolated vertices.

Since one of $H_1 - F_1$ and $H_2 - F_2$ has isolated vertices, without loss of generality, let it be $H_1 - F_1$. According to Proposition 7.1.2, two vertices have at most three common neighbor vertices. Note that $2 \cdot \frac{(n-1)(n-2)}{2} - 3 = (n-1)^2 - (n-1) - 3$. Then $H_1 - F_1$ has at most two isolated vertices. Suppose that there are two isolated vertices in $H_1 - F_1$, let them be a and b. Since F is a nature cut of CK_n, there is no isolated vertex in $CK_n - F$ and neither a nor b is the isolated vertex in $CK_n - F$. Note that $|F_1| = (n-1)^2 - (n-1) - 3$. Since $|F| - |F_1| - |F_2| \leq n^2 - n - 3 - (n-1)^2 + (n-1) + 3 - \frac{(n-1)(n-2)}{2} < (n-2)$ and $|F_1| \geq |F_2| \geq \ldots \geq |F_n|$, we have $|F_n| = 0$. Since each vertex of $CK_n[\bigcup_{i=1}^{n-1} V(H_i - F_i)]$ is adjacent to one vertex in $H_n - F_n = H_n$, $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_n. Then $H_1 - F_1$ has at most one isolated vertex. If there is only one isolated vertex in $H_1 - F_1$, let it be a and the components in $H_1 - F_1 - a$ be $G_1, G_2, \ldots, G_k$ for $k \geq 1$. Since F is a nature cut of CK_n, a is adjacent to one vertex in $H_i - F_i$ for at least one $i \in \{2, \ldots, n\}$. For G_r $(1 \leq r \leq k)$, we have $|V(G_r)| \geq 2$. Since $(n^2 - n - 3 - 2 \cdot \frac{(n-1)(n-2)}{2}) = 2n - 5 < 2(n-2) \leq |N(V(G_r)) \setminus (V(H_1) \cup V(H_2))|$, we have $CK_n[V(G_r) \cup V(H_i - F_i)]$ is connected for at least one $i \in \{3, \ldots, n\}$. The cases of other components in $H_1 - F_1$ are similar. Similarly to the proof of Case 1, we have $CK_n[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected for $i, j \in \{3, \ldots, n\}$. Similarly to the proof of Case 2.1.1, $CK_n[V(H_2 - F_2) \cup V(H_i - F_i)]$ is connected for at least one $i \in \{3, \ldots, n\}$. Therefore, $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_n.

Case 2.1.3. Each of $\{H_1 - F_1, H_2 - F_2\}$ has isolated vertices.

Suppose that $H_1 - F_1$ has two isolated vertices. Then $|F_1| = (n-1)^2 + (n-1) + 3$. Since $|F| - |F_1| - |F_2| \leq n^2 - n - 3 - (n-1)^2 + (n-1) + 3 - \frac{(n-1)(n-2)}{2} < (n-2)$ and $|F_1| \geq |F_2| \geq \ldots \geq |F_n|$, we have $|F_n| = 0$. Since each vertex of $CK_n[\bigcup_{i=1}^{n-1} V(H_i - F_i)]$ is adjacent to one vertex in $H_n - F_n = H_n$, we have $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_n. Then $H_1 - F_1$ has one isolated vertex. If $H_2 - F_2$ has two isolated vertices, similarly we have $|F_n| = 0$. Since each vertex of $CK_n[\bigcup_{i=1}^{n-1} V(H_i - F_i)]$ is adjacent to one vertex in $H_n - F_n = H_n$, we have $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_n. Then each of $\{H_1 - F_1, H_2 - F_2\}$ has one isolated vertex. Let them be a and b. Let the components in $H_1 - F_1 - a$ be $G_1^1, G_2^1, \ldots, G_k^1$ for $k \geq 1$ and in $H_2 - F_2 - b$ be $G_1^2, G_2^2, \ldots, G_l^2$ for $l \geq 1$. Then we have $|V(G_r^1)| \geq 2$ and $|V(G_s^2)| \geq 2$ for $1 \leq r \leq k$ and $1 \leq s \leq l$. Note that there is no isolated vertex in $CK_n - F$. If a is not adjacent to b, then a is adjacent to one vertex in $H_i - F_i$ for at least one $i \in \{3, \ldots, n\}$ or one vertex in G_s^2 for one s $(1 \leq s \leq l)$, and b is adjacent to one vertex in $H_i - F_i$ for at least one $i \in \{3, \ldots, n\}$ or one vertex in G_r^1 for one r $(1 \leq r \leq k)$. For G_r^1, since $(n^2 - n - 3 - 2 \cdot \frac{(n-1)(n-2)}{2}) = 2n - 5 < 2(n-2) \leq |N(V(G_r)) \setminus (V(H_1) \cup V(H_2))|$, we have $CK_n[V(G_r^1) \cup V(H_i - F_i)]$ is connected for at least one $i \in \{3, \ldots, n\}$. Similarly, $CK_n[V(G_s^2) \cup V(H_i)]$ is connected for at least one $i \in \{3, \ldots, n\}$. The cases of other components in $H_1 - F_1$ and $H_2 - F_2$ are similar. Similarly to the proof of Case 1, we have that $CK_n[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected for $i, j \in \{3, \ldots, n\}$. Therefore, $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_n. Suppose that a is adjacent to b. Similarly to the proof before, $CK_n[V(G_r^1) \cup V(H_i - F_i)]$ and $CK_n[V(G_s^2) \cup V(H_i - F_i)]$ are connected for at least one $i \in \{3, \ldots, n\}$ and $CK_n[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected for $i, j \in \{3, \ldots, n\}$. The cases of other components in $H_1 - F_1$ and $H_2 - F_2$ are similar. To cut all the cross-edges of $CK_n[\{a, b\}]$, we need $2(n-2)$ faulty vertices in H_i's, where $i \in \{3, \ldots, n\}$. Since $|F \setminus (F_1 \cup F_2)| \leq n^2 - n - 3 - (n-1)(n-2) = 2n - 5$. Then we have $CK_n[V(H_i - F_i) \cup \{a, b\}]$ is connected for at least one $i \in \{3, \ldots, n\}$. Therefore, $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_n.

Case 2.2. $|F_2| < \frac{(n-1)(n-2)}{2}$.

Case 2.2.1. $H_1 - F_1$ has no isolated vertex.

By the inductive hypothesis, $H_1 - F_1$ is connected. Similarly to the proof of Case 2.1.1, we have that $CK_n[V(H_1 - F_1) \cup V(H_i - F_i)]$ for at least one $i \in \{2, \ldots, n\}$. Similarly to the proof of Case 1, we have $CK_n[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected for $i, j \in \{2, \ldots, n\}$. Therefore, $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_n.

Case 2.2.2. $H_1 - F_1$ has isolated vertices.

According to Proposition 7.1.2, two vertices have at most three common neighbor vertices. Note that $2 \cdot \frac{(n-1)(n-2)}{2} - 3 = (n-1)^2 - (n-1) - 3$. Then $H_1 - F_1$ has at most two isolated vertices. Suppose that $H_1 - F_1$ has two isolated vertices, a and b. Since there is no isolated vertex in $CK_n - F$, we have that neither a nor b is the isolated vertex in $CK_n - F$. Then each of $\{a, b\}$ is adjacent to at least one vertex of $H_i - F_i$, where $i \in \{2, \ldots, n\}$. Let the components in $H_1 - F_1 - a - b$ be $G_1, G_2, \ldots, G_k$ for $k \geq 1$. We have $|V(G_r)| \geq 2$ for $1 \leq r \leq k$. Let $|V(G_1)| = 2$. Since $|N_{H_1}(V(G_1))| = (n-1)(n-2) - 2 = n^2 - 3n > (n-1)^2 - (n-1) - 3$, a contradiction. Therefore, we have $|V(G_r)| \geq 3$. Since $3(n-1) > n^2 - n - 3 - [(n-1)^2 - (n-1) - 3]$, we have $CK_n[V(G_r) \cup V(H_i - F_i)]$ is connected for at least one $i \in \{2, \ldots, n\}$. The cases of other components in $H_1 - F_1$ are similar. Similarly to the proof of Case 1, we have that $CK_n[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected for $i, j \in \{2, \ldots, n\}$. Therefore, $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_n. Then there is only one isolated vertex in $H_1 - F_1$, let it be a and the components in $H_1 - F_1 - a$ be $G_1, G_2, \ldots, G_k$ for $k \geq 1$. Note that $CK_n - F$ has no isolated vertex. Then a is adjacent to one vertex in $H_i - F_i$ for at least one $i \in \{2, \ldots, n\}$. For G_r we have $|V(G_r)| \geq 2$, where $1 \leq r \leq k$. Let $|V(G_1)| = 2$. Since $|N_{H_1}(V(G_1))| = (n-1)(n-2) - 2 = n^2 - 3n > (n-1)^2 - (n-1) - 3$, a contradiction. Therefore, we have $|V(G_r)| \geq 3$. Since $3(n-1) > n^2 - n - 3 - [(n-1)^2 - (n-1) - 3]$, we have $CK_n[V(G_r) \cup V(H_i - F_i)]$ is connected for at least one $i \in \{2, \ldots, n\}$. The cases of other components in $H_1 - F_1$ are similar. Similarly to the proof of Case 1, we have that $CK_n[V(H_i - F_i) \cup V(H_j - F_j)]$ is connected for $i, j \in \{2, \ldots, n\}$. Therefore, $CK_n - F$ is connected, a contradiction to that F is a nature cut of CK_n.

Case 3. $(n-1)^2 - (n-1) - 3 < |F_1| \leq n^2 - n - 3$.

Since $2[(n-1)^2 - (n-1) - 3] = 2n^2 - 6n - 2 > n^2 - n - 3$ for $n \geq 6$, there is only one F_1 such that $(n-1)^2 - (n-1) - 3 \leq |F_1| \leq n^2 - n - 3$.

For other F_i's ($i \in \{2,\dots,n\}$), since $|F-F_1| < n^2-n-3-(n^2-3n-1) = 2n-2$, i.e., $|F-F_1| \le 2n-3$, we have $2n-3 < \frac{(n-1)(n-2)}{2}$ for $n \ge 6$. Therefore, there is no F_i such that $\frac{(n-1)(n-2)}{2} \le |F_i| \le (n-1)^2-(n-1)-3$, where $i \in \{2,\dots,n\}$ and $n \ge 6$. Thus, we have $|F_i| < \frac{(n-1)(n-2)}{2}$ for every $i \in \{2,\dots,n\}$.

Similarly to the proof of Case 1, we have $CK_n[\bigcup_{i=2}^{n} V(H_i-F_i)]$ is connected. Suppose that H_1-F_1 is connected. Since $|E_{1,i}(CK_n)| = (n-1)! > n^2-n-3+\frac{(n-1)(n-2)}{2} \ge |F_1|+|F_i|$ for $i \in \{2,\dots,n\}$, we have $CK_n[V(H_1-F_1)\cup V(H_n-F_n)]$ is connected and hence CK_n-F is connected, a contradiction to that F is a nature cut of CK_n. Then suppose that F_1 is a nature cut of H_1, let the components in H_1-F_1 be $G_1,G_2,\dots,G_k$ for $k \ge 1$. Note that $|V(G_r)| \ge 2$ for $1 \le r \le k$. Then the number of cross-edges for each G_r are at least $2(n-1)$. Note that $|F-F_1| \le 2n-3 < 2(n-1)$, we have $CK_n[V(G_r)\cup V(H_i-F_i)]$ is connected for at least one $i \in \{2,\dots,n\}$. The cases of other components in H_1-F_1 are similar. Therefore, CK_n-F is connected, a contradiction to that F is a nature cut of CK_n. If H_1-F_1 has isolated vertices, let them be $\{v_1,\dots,v_t\}$, where $t \ge 1$. Since $2\cdot\frac{n(n-1)}{2}-3 = n^2-n-3$, there are at most two isolated vertices in CK_n-F. Note there is no isolated vertex in CK_n-F. Then each isolated vertex in H_1-F_1 is adjacent to one vertex of H_i-F_i for at least one $i \in \{2,\dots,n\}$. Let the components in $H_1-F_1-\bigcup_{i=1}^{t} v_i$ be $G_1,G_2,\dots,G_k$ for $k \ge 1$. Then we have $|V(G_r)| \ge 2$, where $1 \le r \le k$. Since $|F|-|F_1| \le n^2-n-3-(n-1)^2+(n-1)+3-1 = 2n-3 < 2(n-1) \le |N(V(G_r))\setminus(V(H_1)|$, we have $CK_n[V(G_r)\cup V(H_i-F_i)]$ is connected for at least one $i \in \{2,\dots,n\}$. The cases of other components in H_1-F_1 are similar. Similarly to the proof of Case 1, we have $CK_n[V(H_i-F_i)\cup V(H_j-F_j)]$ is connected for $i,j \in \{2,\dots,n\}$. Therefore, CK_n-F is connected, a contradiction to that F is a nature cut of CK_n.

By Cases 1–3, F is not a nature cut of CK_n if $|F| \le n^2-n-3$. Therefore, $|F| \ge n^2-n-2$ if F is a nature cut of CK_n. Combining this with $K^{(1)}(CK_n) \le n^2-n-2$, we have $\kappa^*(CK_n) = n^2-n-2$. □

7.4 The Nature Connectivity and Diagnosability of Cayley Graphs Generated by Complete Graphs under PMC Model

In this section, we will study the nature diagnosability of the Cayley graph CK_n generated by the complete graph K_n under the PMC model.

Firstly we give an important lemma which will be used in the proof to determine the nature diagnosability of CK_n under PMC Model, where $n \geq 4$.

Lemma 7.4.1 Let $A = \{(1),(12)\}$ and CK_n be defined as above. If $n \geq 4$, $F_1 = N_{CK_n}(A)$, $F_2 = A \cup N_{CK_n}(A)$, then $|F_1| = n^2 - n - 2$, $|F_2| = n^2 - n$, $\delta(CK_n - F_1) \geq 1$, and $\delta(CK_n - F_2) \geq 1$.

Proof: By $A = \{(1),(12)\}$, we have $CK_n[A] \cong CK_2 = K_2$. Since CK_n has no 3-cycles, $|N_{CK_n}(A)| = n^2 - n - 2$. Thus from calculating, we have $|F_1| = n^2 - n - 2$, $|F_2| = |A| + |F_1| = n^2 - n$.

In F_1 we will prove at most three vertices which are adjacent to one vertex x in $S_n \setminus F_2$, i.e., $|N_{CK_n}(x) \cap F_2)| \leq 3$ for any $x \in S_n \setminus F_2$. Note that $CK_n - F_1$ has two parts $CK_n - F_2$ and CK_2 (for convenience). Since $F_1 = N_{CK_n}(A)$, x is not adjacent to each vertex of $V(CK_2) = A$. If $|N(x) \cap N((1))| \neq 0$, then $|N(x) \cap N((12))| = 0$ by Theorem 7.1.1. By Proposition 7.1.2, we have that $|N(x) \cap N((1))| \leq 3$. Therefore, $\delta(CK_n - F_2) \geq \frac{n(n-1)}{2} - 3$. $CK_n - F_1$ has two parts $CK_n - F_2$ and CK_2 (for convenience). Note that $\delta(CK_2) = 1$. When $n \geq 4$, we have that $\delta(CK_n - F_2) \geq \frac{n(n-1)}{2} - 3 \geq 1$. Therefore, $\delta(CK_n - F_1) \geq 1$ for $n \geq 4$. □

Secondly we give the upper bound of the nature diagnosability of the Cayley graph CK_n generated by the complete graph K_n under the PMC model.

Lemma 7.4.2 Let $n \geq 4$. Then the nature diagnosability of the Cayley graph CK_n generated by the complete graph K_n under the PMC model is less than or equal to $n^2 - n - 1$, i.e., $t_1(CK_n) \leq n^2 - n - 1$.

Proof: Let A be defined as above, and let $F_1 = N_{CK_n}(A)$, $F_2 = A \cup N_{CK_n}(A)$ (See Fig. 5.2). By Lemma 7.4.1, $|F_1| = n^2 - n - 2$, $|F_2| = n^2 - n$, $\delta(CK_n - F_1) \geq 1$ and $\delta(CK_n - F_2) \geq 1$. Therefore, F_1 and F_2 are both nature faulty sets of CK_n with $|F_1| = n^2 - n - 2$ and $|F_2| = n^2 - n$. Since $A = F_1 \triangle F_2$ and $N_{CK_n}(A) = F_1 \subset F_2$, there is no edge of CK_n between $V(CK_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 5.2.2, we can see that CK_n is not nature $(n^2 - n)$-diagnosable under PMC model. Hence, by the definition of the nature diagnosability, we conclude that the nature diagnosability of CK_n is less than $(n^2 - n)$, i.e., $t_1(CK_n) \leq n^2 - n - 1$. □

Thirdly we prove the lower bound of the nature diagnosability of the Cayley graph CK_n generated by the complete graph K_n under the PMC model.

Lemma 7.4.3 Let $n \geq 4$. Then the nature diagnosability of the Cayley graph CK_n generated by the complete graph K_n under the PMC model is more than or equal to $n^2 - n - 1$, i.e., $t_1(CK_n) \geq n^2 - n - 1$.

Proof: By the definition of the nature diagnosability, it is sufficient to show that CK_n is nature $(n^2 - n - 1)$-diagnosable. By Theorem 5.2.2, to prove CK_n is nature $(n^2 - n - 1)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(CK_n)$ with $u \in V(CK_n) \setminus (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of nature faulty subsets F_1 and F_2 of $V(CK_n)$ with $|F_1| \leq n^2 - n - 1$ and $|F_2| \leq n^2 - n - 1$.

We prove this by contradiction. Suppose that there are two distinct nature faulty subsets F_1 and F_2 of $V(CK_n)$ with $|F_1| \leq n^2 - n - 1$ and $|F_2| \leq n^2 - n - 1$, but the vertex set pair (F_1, F_2)does not satisfy the condition in Theorem 5.2.2, i.e., there are no edges between $V(CK_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Suppose $V(CK_n) = F_1 \cup F_2$. By the definition of CK_n, $|F_1 \cup F_2| = |S_n| = n!$. It is obvious that $n! > 2(n^2 - n - 1)$ for $n \geq 4$.Since $n \geq 4$, we have that $n! = |V(CK_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(n^2 - n - 1)$, a contradiction.Therefore, $V(CK_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(CK_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a nature faulty set, $CK_n - F_1$ has two parts $CK_n - F_1 - F_2$ and $CK_n[F_2 \setminus F_1]$ (for convenience). Thus, $\delta(CK_n - F_1 - F_2) \geq 1$ and $\delta(CK_n[F_2 \setminus F_1]) \geq 1$. Similarly, $\delta(CK_n[F_1 \setminus F_2]) \geq 1$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a nature faulty set. Since there are no edges between

$V(CK_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a nature cut. Since $n \geq 4$, by Theorem 7.3.3, $|F_1 \cap F_2| \geq n^2 - n - 2$. Since $\delta(CK_n[F_2 \setminus F_1]) \geq 1$, $|F_2 \backslash F_1| \geq 2$ holds. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq 2 + n^2 - n - 2 = n^2 - n$, which contradicts with that $|F_2| \leq n^2 - n - 1$. Let $F_1 \setminus F_2 = \emptyset$. Then $F_1 \subseteq F_2$. Since F_1 is a nature set of CK_n, we have $\delta(CK_n[F_2 \setminus F_1]) \geq 1$ and $\delta(CK_n - F_1 - F_2) \geq 1$. Since there are no edges between $V(CK_n - F_1 - F_2)$ and $CK_n[F_2 \setminus F_1]$, we have that F_1 is a nature cut of CK_n. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| = |F_2 \backslash F_1| + |F_1| \geq 2 + n^2 - n - 2 = n^2 - n$, which contradicts with that $|F_2| \leq n^2 - n - 1$. So CK_n is nature $(n^2 - n - 1)$-diagnosable. By the definition of $t_1(CK_n)$, $t_1(CK_n) \geq n^2 - n - 1$. □

Combining Lemma 7.4.2 and 7.4.3, we have the following theorem.

Theorem 7.4.4 Let $n \geq 4$. Then the nature diagnosability of the Cayley graph CK_n generated by the complete graph K_n under PMC model is $n^2 - n - 1$.

7.5 The Nature Connectivity and Diagnosability of Cayley Graphs Generated by Complete Graphs under the MM* Model

In this section, we will study the nature diagnosability of the Cayley graph CK_n generated by the complete graph K_n under the MM* model.

Firstly we give the upper bound of its nature diagnosability under the MM* model.

Lemma 7.5.1 Let $n \geq 4$. Then the nature diagnosability of the Cayley graph CK_n generated by the complete graph K_n under the MM* model is less than or equal to $n^2 - n - 1$, i.e., $t_1(CK_n) \leq n^2 - n - 1$.

Proof: Let A, F_1 and F_2 be defined in Lemma 7.4.1 (See Fig. 5.2). By Lemma 7.4.1, $|F_1| = n^2 - n - 2$, $|F_2| = n^2 - n$, $\delta(CK_n - F_1) \geq 1$ and $\delta(CK_n - F_2) \geq 1$. So both F_1 and F_2 are nature faulty sets. By the definitions of F_1 and F_2, $F_1 \triangle F_2 = A$. Note $F_1 \setminus F_2 = \emptyset$, $F_2 \setminus F_1 = A$ and $(V(CK_n) \setminus (F_1 \cup F_2)) \cap A = \emptyset$. Therefore, both F_1 and F_2 does not satisfy any one condition in Theorem 5.3.2, and CK_n is not nature (n^2-n)-diagnosable. Hence, $t_1(CK_n) \le n^2-n-1$. The proof is completed. □

Then we prove the lower bound of the nature diagnosability of CK_n under the MM* model.

Lemma 7.5.2 Let $n \ge 5$. Then the nature diagnosability of the Cayley graph CK_n generated by the complete graph K_n under the MM* model is more than or equal to n^2-n-1, i.e., $t_1(CK_n) \ge n^2-n-1$.

Proof: By the definition of the nature diagnosability, it is sufficient to show that CK_n is nature (n^2-n-1)-diagnosable.

By Theorem 5.3.2, suppose, on the contrary, that there are two distinct nature faulty subsets F_1 and F_2 of CK_n with $|F_1| \le n^2-n-1$ and $|F_2| \le n^2-n-1$, but the vertex set pair (F_1,F_2) does not satisfy any one condition in Theorem 5.3.2. Without loss of generality, assume that $F_2 \setminus F_1 \ne \emptyset$. Similarly to the discussion on $V(CK_n) \ne F_1 \cup F_2$ in Lemma 7.4.3, we can deduce $V(CK_n) \ne F_1 \cup F_2$. Therefore, $V(CK_n) \ne F_1 \cup F_2$.

Claim I. $CK_n - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $CK_n - F_1 - F_2$ has at least one isolated vertex w. Since F_1 is a nature faulty set, there is a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Since the vertex set pair (F_1,F_2) does not satisfy any condition in Theorem 5.3.2, there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Thus, there is just one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Similarly, we can deduce that there is just one vertex $v \in F_1 \setminus F_2$ such that v is adjacent to w when $F_1 \setminus F_2 \ne \emptyset$. Let $W \subseteq S_n \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $CK_n[S_n \setminus (F_1 \cup F_2)]$, and let H be the subgraph induced by the vertex set $S_n \setminus (F_1 \cup F_2 \cup W)$. Then for any $w \in W$, there are $\frac{n(n-1)}{2}-2$ neighbors in $F_1 \cap F_2$ when $F_1 \setminus F_2 \ne \emptyset$. Since $|F_2| \le n^2-n-1$, we have $\sum_{w\in W} |N_{CK_n[(F_1\cap F_2)\cup W]}(w)| = |W|(\frac{n(n-1)}{2}-2) \le \sum_{v\in F_1\cap F_2} d_{CK_n}(v) = |F_1 \cap F_2|\frac{n(n-1)}{2} \le (|F_2|-1)\frac{n(n-1)}{2} \le \frac{(n-2)(n-1)n(n+1)}{2}$. It follows that $|W| \le \frac{(n-2)(n-1)n(n+1)}{n^2-n-4} \le \frac{(n-2)(n-1)n(n+1)}{(n-2)(n-1)} = n(n+1)$ for $n \ge 5$. Note $|F_1 \cup F_2| = |F_1|+|F_2|-|F_1 \cap F_2| \le 2(n^2-n-1) - \frac{n(n-1)}{2} + 2 = \frac{3}{2}n(n-1)$. Suppose $V(H) = \emptyset$. Then $n! = |S_n| = |V(CK_n)| = |F_1 \cup F_2| + |W| \le \frac{3}{2}n(n-1) + n(n+1)$. This is a contradiction to $n \geq 5$. So $V(H) \neq \emptyset$ when $n \geq 5$. Since the vertex set pair (F_1, F_2)does not satisfy the condition (1) of Theorem 5.3.2, and any vertex of $V(H)$ is not isolated in H, we deduce that there is no edge between $V(H)$ and $F_1 \triangle F_2$. Thus, $F_1 \cap F_2$ is a vertex cut of CK_n and $\delta(CK_n - (F_1 \cap F_2)) \geq 1$, i.e., $F_1 \cap F_2$ is a nature cut of CK_n. By Theorem 7.3.3, $|F_1 \cap F_2| \geq n^2 - n - 2$. Because $|F_1| \leq n^2 - n - 1$, $|F_2| \leq n^2 - n - 1$, and neither $F_1 \setminus F_2$ nor $F_2 \setminus F_1$ is empty, we have $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$. Let $F_1 \setminus F_2 = \{v_1\}$ and $F_2 \setminus F_1 = \{v_2\}$. Then for any vertex $w \in W$, w is adjacent to v_1 and v_2. According to Proposition 7.1.2, there are at most three common neighbors for any pair of vertices in CK_n, it follows that there are at most three isolated vertices in $CK_n - F_1 - F_2$.

Suppose that there is exactly one isolated vertex v in $CK_n - F_1 - F_2$. Let v_1 and v_2 be adjacent to v. Then $N_{CK_n}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$. Since CK_n contains no triangle, it follows that $N_{CK_n}(v_j) \setminus \{v\} \subseteq F_1 \cap F_2$ and $[N_{CK_n}(v) \setminus \{v_1, v_2\}] \cap [N_{CK_n}(v_j) \setminus \{v\}] = \emptyset$ for $j \in \{1,2\}$. By Proposition 7.1.2, there are at most three common neighbors for any pair of vertices in CK_n. Thus, it follows that $|\bigcap_{j=1}^{2}[N_{CK_n}(v_j) \setminus \{v\}]| \leq 2$. Thus, $|F_1 \cap F_2| \geq |N_{CK_n}(v) \setminus \{v_1, v_2\}| + \sum_{j=1}^{2} |N_{CK_n}(v_j) \setminus \{v\}| - |\bigcap_{j=1}^{2}[N_{CK_n}(v_j) \setminus \{v\}]| = \frac{n(n-1)}{2} - 2 + 2(\frac{n(n-1)}{2} - 1) - 2 = \frac{3}{2}n(n-1) - 6$. Since $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + \frac{3}{2}n(n-1) - 6 = \frac{3}{2}n(n-1) - 5 > n^2 - n - 1$, where $n \geq 5$, which contradicts $|F_2| \leq n^2 - n - 1$.

Suppose that there are exactly two isolated vertices v'_1 and v'_2 in $CK_n - F_1 - F_2$. Let v_1 and v_2 be adjacent to v'_1 and v'_2, respectively. Then $N_{CK_n}(v'_i) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$ for $i \in \{1,2\}$. Since CK_n contains no triangle, it follows that $N_{CK_n}(v_j) \setminus \{v'_1, v'_2\} \subseteq F_1 \cap F_2$, $[N_{CK_n}(v'_i) \setminus \{v_1, v_2\}] \cap [N_{CK_n}(v_j) \setminus \{v'_1, v'_2\}] = \emptyset$, where $i, j \in \{1,2\}$. By Proposition 7.1.2, there are at most three common neighbors for any pair of vertices in CK_n. Thus, it follows that $|\bigcap_{j=1}^{2}[N_{CK_n}(v_j) \setminus \{v'_1, v'_2\}]| = 1$. Thus, $|F_1 \cap F_2| \geq \sum_{i=1}^{2} |N_{CK_n}(v'_i) \setminus \{v_1, v_2\}| + \sum_{j=1}^{2} |N_{CK_n}(v_j) \setminus \{v'_1, v'_2\}| - |\bigcap_{j=1}^{2}[N_{CK_n}(v_j) \setminus \{v'_1, v'_2\}]| = 4(\frac{n(n-1)}{2} - 2) - 1 = 2n(n-1) - 9$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 2n(n-1) - 8 > n^2 - n - 1$ for $n \geq 5$, which contradicts $|F_2| \leq n^2 - n - 1$.

Suppose that there are exactly three isolated vertices v'_i in $CK_n - F_1 - F_2$ for $i \in \{1,2,3\}$. Let v_1 and v_2 be adjacent to v'_i, respectively. Then $N_{CK_n}(v'_i) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$. Since CK_n contains no triangle, it follows that $N_{CK_n}(v_j) \setminus \{v'_1, v'_2, v'_3\} \subseteq F_1 \cap F_2$, $[N_{CK_n}(v'_i) \setminus \{v_1, v_2\}] \cap$

$[N_{CK_n}(v_j) \setminus \{v'_1, v'_2, v'_3\}] = \emptyset$, where $i \in \{1,2,3\}$ and $j \in \{1,2\}$. By Proposition 7.1.2, there are at most three common neighbors for any pair of vertices in CK_n. Thus, it follows that $|\bigcap_{j=1}^{2}[N_{CK_n}(v_j) \setminus \{v'_1, v'_2, v'_3\}]| = 0$. Thus, $|F_1 \cap F_2| \geq \sum_{i=1}^{3} |N_{CK_n}(v'_i) \setminus \{v_1, v_2\}| + \sum_{j=1}^{2} |N_{CK_n}(v_j) \setminus \{v'_1, v'_2, v'_3\}| = \frac{5}{2}n(n-1) - 12$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq \frac{5}{2}n(n-1) - 11 > n^2 - n - 1$ for $n \geq 5$, which contradicts $|F_2| \leq n^2 - n - 1$.

Suppose $F_1 \setminus F_2 = \emptyset$. Then $F_1 \subseteq F_2$. Since F_2 is a nature faulty set, $CK_n - F_2 = S_n - F_1 - F_2$ has no isolated vertex. The proof of Claim is completed.

Let $u \in V(CK_n) \setminus (F_1 \cup F_2)$. By Claim I, u has at least one neighbor in $CK_n - F_1 - F_2$. Since the vertex set pair (F_1, F_2) does not satisfy any one condition in Theorem 5.3.2, by the condition (1) of Theorem 5.3.2, for any pair of adjacent vertices $u, w \in V(CK_n) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \bigtriangleup F_2$ such that $uw \in E(CK_n)$ and $vw \in E(CK_n)$. It follows that u has no neighbor in $F_1 \bigtriangleup F_2$. By the arbitrariness of u, there is no edge between $V(CK_n) \setminus (F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a nature faulty set, $\delta_{CK_n}([F_2 \setminus F_1]) \geq 1$. Since $\delta(CK_n[F_2 \setminus F_1]) \geq 1$, $|F_2 \setminus F_1| \geq 2$ holds. Since both F_1 and F_2 are nature faulty sets, and there is no edge between $V(CK_n) \setminus (F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$, $F_1 \cap F_2$ is a nature cut of CK_n. By Lemma 7.3.3, we have $|F_1 \cap F_2| \geq n^2 - n - 2$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 2 + n^2 - n - 2 = n^2 - n$, which contradicts $|F_2| \leq n^2 - n - 1$. Therefore, CK_n is nature $n^2 - n - 1$-diagnosable and $t_1(CK_n) \geq n^2 - n - 1$. The proof is completed. □

Combining Lemma 7.5.1 and 7.5.2, we have the following theorem.

Theorem 7.5.3 Let $n \geq 5$. Then the nature diagnosability of the Cayley graph CK_n generated by the complete graph K_n under MM* model is $n^2 - n - 1$.

7.6 Conclusion

In this chapter, we investigated the problem of the nature diagnosability of CK_n under the PMC model and MM* model. It is proved that the nature connectivity of CK_n is $n^2 - n - 2$ and the nature diagnosability of CK_n under the PMC model is $n^2 - n - 1$ for $n \geq 4$ and under the MM* model is $n^2 - n - 1$ for $n \geq 5$. Note that CK_n and $C\Gamma_n$ are both generated by transpositions. However, since tree is the subgraph of complete graph, the results on CK_n are more general.

Chapter 8

The Nature Diagnosability of Bubble-Sort Star Graph under the PMC Model & MM* Model

In this chapter, we show that the nature diagnosability of BS_n is $4n-7$ under the PMC model for $n \geq 4$, the nature diagnosability of BS_n is $4n-7$ under the MM* model for $n \geq 5$. The results in this chapter is published in International Journal of Engineering and Applied Sciences [86].

8.1 Background & Known Results

As we defined in chapter 2, Bubble-sort Star graph BS_n is Cayley graph generated by transpositions, thus we have Proposition 8.1.1 and Proposition 8.1.2.

Proposition 8.1.1 For any integer $n \geq 1$, BS_n is $(2n-3)$-regular, vertex-transitive.

Proposition 8.1.2 For any integer $n \geq 2$, BS_n is bipartite.

Since the generating set of BS_n contains two transpositions, which are disjoint, it is straightforward to prove the following Proposition 8.1.3.

Proposition 8.1.3 For any integer $n \geq 3$, the girth of BS_n is 4.

By Theorem 2.4.3, a simple connected graph H can be labelled properly. We can partition BS_n into n subgraphs $BS_1, BS_2, \ldots, BS_n$, where every vertex $u = x_1x_2\ldots x_n \in V(BS_n)$ has a fixed integer i in the last position x_n for $i \in [n]$. It is obvious that BS_n^i is isomorphic to BS_{n-1} for $i \in [n]$. Let $v \in V(BS_n^i)$, then $v(1n)$ and $v(n-1,n)$ are called outside neighbors of v.

The following two propositions are from [15].

Proposition 8.1.4 [15] Let BS_n^i be defined as above. There are $2(n-2)!$ independent cross-edges between two different H_i's.

Proposition 8.1.5 [15] Let BS_n be the bubble-sort star graph. If two vertices u, v are adjacent, there is no common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| = 0$. If two vertices u, v are not adjacent, there are at most three common neighbor vertices of these two vertices, i.e., $|N(u) \cap N(v)| \leq 3$.

Next we include results on the nature connectivity of BS_n, which is a indispensable part combined with Lemma 7.4.1 in proof to determine the nature diagnosability of $C\Gamma_n$ under PMC Model or MM*, where $n \geq 4$.

Lemma 8.1.1 [97] The nature connectivity $\kappa^*(BS_4)$ of the bubble-sort star graph BS_4 is 8.

Theorem 8.1.2 [96] For $n \geq 5$, the bubble-sort star graph BS_n is tightly $(4n-8)$ super-nature-connected.

To show the nature diagnosability of Bubble-sort star graph under the PMC model, we shall first prove the following Lemma.

Lemma 8.1.3 Let $A = \{(1), (12)\}$. If $n \geq 4$, $F_1 = N_{BS_n}(A)$, $F_2 = A \cup N_{BS_n}(A)$, then $|F_1| = 4n-8$, $|F_2| = 4n-6$, $\delta(BS_n - F_1) \geq 1$, and $\delta(BS_n - F_2) \geq 1$.

Proof: Since $A = \{(1), (12)\}$, we have $BS_n[A] \cong BS_2 = K_2$. Since BS_n has not 3-cycles, we have $|N_{BS_n}(A)| = 4n-8$. Thus from calculating, we have $|F_1| = 4n-8$, $|F_2| = |A| + |F_1| = 4n-6$.

Claim 1 . For any $x \in S_n \setminus F_2$, $|N_{BS_n}(x) \cap F_2)| \leq 2n-4$.

Since BS_n is a bipartite graph, there is no 5-cycle $(1),(ki),x,(12)(lj),(12),(1)$ of BS_n, where $(ki),(lj) \in S \setminus (12)$. Let $u \in N_{BS_n}((1)) \setminus (12)$. If u is adjacent to x, then x is not adjacent to each of $N_{BS_n}((12)) \setminus (1)$. Since $|N_{BS_n}((1)) \setminus (12)| = 2n-4$, we have that x is adjacent to at most $(2n-4)$ vertices in F_1.

By Claim 1, $|N_{BS_n}(x) \cap F_2)| \leq 2n-4$ for any $x \in S_n \setminus F_2$. Therefore, $\delta(BS_n - F_2) \geq 2n-3-(2n-4) = 1$. $BS_n - F_1$ has two components $BS_n - F_2$ and BS_2. Note that $\delta(BS_2) = 1$, therefore, $\delta(BS_n - F_1) \geq 1$. □

8.2 The Nature Diagnosability of Bubble-Sort Star Graph under the PMC Model

Let F_1 and F_2 be two distinct subsets of V for a system $G = (V,E)$. Define the symmetric difference $F_1 \triangle F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. Yuan et al. [112] presented a sufficient and necessary condition for a system to be nature t-diagnosable under the PMC model. See Theorem 5.2.2.

Lemma 8.2.1 A graph of minimum degree 1 has at least two vertices.

The proof of Lemma 8.2.1 is straightforward.

Lemma 8.2.2 Let $n \geq 4$. Then the nature diagnosability of the bubble-sort star graph BS_n under the PMC model is less than or equal to $4n-7$, i.e., $t_1(BS_n) \leq 4n-7$.

Proof: Let A be defined in Lemma 7.4.1, and let $F_1 = N_{BS_n}(A)$, $F_2 = A \cup N_{BS_n}(A)$. By Lemma 7.4.1, $|F_1| = 4n-8$, $|F_2| = 4n-6$, $\delta(BS_n - F_1) \geq 1$ and $\delta(BS_n - F_2) \geq 1$. Therefore, F_1 and F_2 are both nature faulty sets of BS_n with $|F_1| = 4n-8$ and $|F_2| = 4n-6$. Since $A = F_1 \triangle F_2$ and $N_{BS_n}(A) = F_1 \subset F_2$, there is no edge of BS_n between $V(BS_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 5.2.2, we know that BS_n is not nature $(4n-6)$-diagnosable under PMC model. Hence, by the definition of nature diagnosability, we conclude that the nature diagnosability of BS_n is less than $4n-6$, i.e., $t_1(BS_n) \leq 4n-7$. □

Lemma 8.2.3 Let $n \geq 4$. Then the nature diagnosability of the bubble-sort star graph BS_n under the PMC model is more than or equal to $4n-7$, i.e., $t_1(BS_n) \geq 4n-7$.

Proof: By the definition of nature diagnosability, it is sufficient to show that BS_n is nature $(4n-7)$-diagnosable. By Theorem 5.2.2, to prove BS_n is nature $(4n-7)$-diagnosable, it is equivalent to show that there is an edge $uv \in E(BS_n)$ with $u \in V(BS_n) \backslash (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of nature faulty subsets F_1 and F_2 of $V(BS_n)$ with $|F_1| \leq 4n-7$ and $|F_2| \leq 4n-7$.

We prove this theorem by contradiction. Suppose that there are two distinct nature faulty subsets F_1 and F_2 of $V(BS_n)$ with $|F_1| \leq 4n-7$ and $|F_2| \leq 4n-7$, but the vertex set pair (F_1, F_2) does not satisfy with the condition in Theorem 5.2.2, i.e., there are no edges between $V(BS_n) \backslash (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Suppose $V(BS_n) = F_1 \cup F_2$. By the definition of BS_n, $|F_1 \cup F_2| = |S_n| = n!$. It is obvious that $n! > 8n-14$ for $n \geq 4$. Since $n \geq 4$, we have that $n! = |V(BS_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(4n-7) = 8n-14$, a contradiction. Therefore, $V(BS_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(BS_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a nature faulty set, $BS_n - F_1$ has two parts $BS_n - F_1 - F_2$ and $BS_n[F_2 \setminus F_1]$ (for convenience). Thus, $\delta(BS_n - F_1 - F_2) \geq 1$ and $\delta(BS_n[F_2 \setminus F_1]) \geq 1$. Similarly, $\delta(BS_n[F_1 \setminus F_2]) \geq 1$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a nature faulty set. When $F_1 \setminus F_2 = \emptyset$, $F_1 \cap F_2 = F_1$ is also a nature faulty set.Since there are no edges between $V(BS_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a nature cut. Since $n \geq 4$, by Theorem 8.1.2, we have $|F_1 \cap F_2| \geq 4n-8$. By Lemma 8.2.1, $|F_2 \backslash F_1| \geq 2$. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq 2 + 4n - 8 = 4n-6$, which contradicts with that $|F_2| \leq 4n-7$. So BS_n is nature $(4n-7)$-diagnosable. By the definition of $t_1(BS_n)$, $t_1(BS_n) \geq 4n-7$. □

Combining Lemmas 8.2.2 and 8.2.3, we have the following theorem.

Theorem 8.2.4 Let $n \geq 4$. Then the nature diagnosability of the bubble-sort star graph BS_n under PMC model is $4n-7$.

8.3 The Nature Diagnosability of Bubble-Sort Star Graph under the MM* model

We firstly present the lower bound of the nature diagnosability of the bubble-sort star graph BS_n under the MM* model.

Lemma 8.3.1 Let $n \geq 4$. Then the nature diagnosability of the bubble-sort star graph BS_n under the MM* model is less than or equal to $4n-7$, i.e., $t_1(BS_n) \leq 4n-7$.

Proof: Let A, F_1 and F_2 be defined in Lemma 8.1.3. By the Lemma 8.1.3, $|F_1| = 4n-8$, $|F_2| = 4n-6$, $\delta(BS_n - F_1) \geq 1$ and $\delta(BS_n - F_2) \geq 1$. So both F_1 and F_2 are nature faulty sets. By the definitions of F_1 and F_2, $F_1 \triangle F_2 = A$. Note $F_1 \setminus F_2 = \emptyset$, $F_2 \setminus F_1 = A$ and $(V(BS_n) \setminus (F_1 \cup F_2)) \cap A = \emptyset$. Therefore, both F_1 and F_2 are not satisfied with any one condition in Theorem 5.3.2, and BS_n is not nature $(3n-6)$-diagnosable. Hence, $t_1(BS_n) \leq 4n-7$. The proof is completed. □

Then we show the upper bound of the nature diagnosability of the bubble-sort star graph BS_n under the MM* model.

Lemma 8.3.2 Let $n \geq 5$. Then the nature diagnosability of the bubble-sort star graph BS_n under the MM* model is more than or equal to $4n-7$, i.e., $t_1(BS_n) \geq 4n-7$.

Proof: By the definition of nature diagnosability, it is sufficient to show that BS_n is nature $(4n-7)$-diagnosable.

By Theorem 5.3.2, suppose, on the contrary, that there are two distinct nature faulty subsets F_1 and F_2 of BS_n with $|F_1| \leq 4n-7$ and $|F_2| \leq 4n-7$, but the vertex set pair (F_1, F_2) does not satisfy any condition in Theorem 5.3.2. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similarly to the discussion on $V(BS_n) \neq F_1 \cup F_2$ in Lemma 8.2.3, we can conclude $V(BS_n) \neq F_1 \cup F_2$. Therefore, $V(BS_n) \neq F_1 \cup F_2$.

Claim 1. $BS_n - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $BS_n - F_1 - F_2$ has at least one isolated vertex w. Since F_1 is a nature faulty set, there is a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Since the vertex set pair (F_1,F_2) does not satisfy any conditions in Theorem 5.3.2, there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Thus, there is just a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Similarly, we can that there is just a single vertex $v \in F_1 \setminus F_2$ such that v is adjacent to w when $F_1 \setminus F_2 \neq \emptyset$. Let $W \subseteq S_n \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $BS_n[S_n \setminus (F_1 \cup F_2)]$, and let H be the subgraph induced by the vertex set $S_n \setminus (F_1 \cup F_2 \cup W)$. Then for any $w \in W$, there are $(2n-5)$ neighbors in $F_1 \cap F_2$. Since $|F_2| \leq 4n-7$, we have $\sum_{w \in W} |N_{BS_n[(F_1 \cap F_2) \cup W]}(w)| = |W|(2n-5) \leq \sum_{v \in F_1 \cap F_2} d_{BS_n}(v) \leq |F_1 \cap F_2|(2n-3) \leq (|F_2|-1)(2n-3) \leq (4n-8)(2n-3) = 8n^2-28n+24$. It follows that $|W| \leq \frac{8n^2-28n+24}{2n-5} < 4n-3$ for $n \geq 5$. Note $|F_1 \cup F_2| = |F_1|+|F_2|-|F_1 \cap F_2| \leq 2(4n-7)-(2n-5) = 6n-9$. Suppose $V(H) = \emptyset$. Then $n! = |S_n| = |V(BS_n)| = |F_1 \cup F_2|+|W| < 6n-9+4n-3 = 10n-11$. This is a contradiction to $n \geq 5$. So $V(H) \neq \emptyset$. Since the vertex set pair (F_1,F_2) does not satisfy the condition (1) of Theorem 5.3.2, and any vertex of $V(H)$ is not isolated in H, we know that there is no edge between $V(H)$ and $F_1 \triangle F_2$. Thus, $F_1 \cap F_2$ is a vertex cut of BS_n and $\delta(BS_n-(F_1 \cap F_2)) \geq 1$, i.e., $F_1 \cap F_2$ is a nature cut of BS_n. By Theorem 8.1.2, we have $|F_1 \cap F_2| \geq 4n-8$. Because $|F_1| \leq 4n-7$, $|F_2| \leq 4n-7$, and neither $F_1 \setminus F_2$ nor $F_2 \setminus F_1$ is empty, we have $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$. Let $F_1 \setminus F_2 = \{v_1\}$ and $F_2 \setminus F_1 = \{v_2\}$. Then for any vertex $w \in W$, w are adjacent to v_1 and v_2. According to Proposition 8.1.4, there are at most three common neighbors for any pair of vertices in BS_n, it follows that there are at most three isolated vertices in $BS_n-F_1-F_2$, i.e., $|W| \leq 3$.

Suppose that there is exactly one isolated vertex v in $BS_n-F_1-F_2$. Let v_1 and v_2 be adjacent to v. Then $N_{BS_n}(v) \setminus \{v_1,v_2\} \subseteq F_1 \cap F_2$. Since BS_n contains no triangle, it follows that $N_{BS_n}(v_1) \setminus \{v\} \subseteq F_1 \cap F_2$; $N_{BS_n}(v_2) \setminus \{v\} \subseteq F_1 \cap F_2$; $[N_{BS_n}(v) \setminus \{v_1,v_2\}] \cap [N_{BS_n}(v_1) \setminus \{v\}] = \emptyset$ and $[N_{BS_n}(v) \setminus \{v_1,v_2\}] \cap [N_{BS_n}(v_2) \setminus \{v\}] = \emptyset$. By Proposition 8.1.4, $|[N_{BS_n}(v_1) \setminus \{v\}] \cap [N_{BS_n}(v_2) \setminus \{v\}]| \leq 2$. Thus, $|F_1 \cap F_2| \geq |N_{BS_n}(v) \setminus \{v_1,v_2\}|+|N_{BS_n}(v_1) \setminus \{v\}|+|N_{BS_n}(v_2) \setminus \{v\}| = (2n-5)+(2n-4)+(2n-4)-2 = 6n-15$. It follows that $|F_2| = |F_2 \setminus F_1|+|F_1 \cap F_2| \geq 1+6n-15 = 6n-14 > 4n-7$ $(n \geq 5)$, which contradicts $|F_2| \leq 4n-7$.

Suppose that there are exactly two isolated vertices v and w in $BS_n-F_1-F_2$. Let v_1 and v_2 be adjacent to v and w, respectively. Then $N_{BS_n}(v) \setminus \{v_1,v_2\} \subseteq F_1 \cap F_2$. Since BS_n contains no triangle, it follows that $N_{BS_n}(v_1) \setminus \{v,w\} \subseteq F_1 \cap F_2$, $N_{BS_n}(v_2) \setminus \{v,w\} \subseteq F_1 \cap F_2$,

$[N_{BS_n}(v)\setminus\{v_1,v_2\}]\cap[N_{BS_n}(v_1)\setminus\{v,w\}]=\emptyset$ and $[N_{BS_n}(v)\setminus\{v_1,v_2\}]\cap[N_{BS_n}(v_2)\setminus\{v,w\}]=\emptyset$. By Proposition 8.1.4, there are at most two common neighbors for any pair of vertices in BS_n. Thus, it follows that $|[N_{BS_n}(v_1)\setminus\{v,w\}]\cap[N_{BS_n}(v_2)\setminus\{v,w\}]|\leq 1$. Thus, $|F_1\cap F_2|\geq |N_{BS_n}(v)\setminus\{v_1,v_2\}|+|N_{BS_n}(w)\setminus\{v_1,v_2\}|+|N_{BS_n}(v_1)\setminus\{v,w\}|+|N_{BS_n}(v_2)\setminus\{v,w\}|=(2n-5)+(2n-5)-1+(2n-5)+(2n-5)-1=8n-22$. It follows that $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\geq 1+8n-22=8n-21>4n-7$ $(n\geq 5)$, which contradicts $|F_2|\leq 4n-7$.

Suppose that there are exactly three isolated vertices u,v and w in $BS_n-F_1-F_2$. Let v_1 and v_2 be adjacent to u,v and w, respectively. Then $N_{BS_n}(v)\setminus\{v_1,v_2\}\subseteq F_1\cap F_2$. Since BS_n contains no triangle, it follows that $N_{BS_n}(v_1)\setminus\{u,v,w\}\subseteq F_1\cap F_2$, $N_{BS_n}(v_2)\setminus\{u,v,w\}\subseteq F_1\cap F_2$, $[N_{BS_n}(v)\setminus\{v_1,v_2\}]\cap[N_{BS_n}(v_1)\setminus\{u,v,w\}]=\emptyset$ and $[N_{BS_n}(v)\setminus\{v_1,v_2\}]\cap[N_{BS_n}(v_2)\setminus\{u,v,w\}]=\emptyset$. By Proposition 8.1.4, there are at most three common neighbors for any pair of vertices in BS_n. Thus, it follows that $|[N_{BS_n}(v_1)\setminus\{u,v,w\}]\cap[N_{BS_n}(v_2)\setminus\{u,v,w\}]|=0$. Thus, $|F_1\cap F_2|\geq|N_{BS_n}(u)\setminus\{v_1,v_2\}|+|N_{BS_n}(v)\setminus\{v_1,v_2\}|+|N_{BS_n}(w)\setminus\{v_1,v_2\}|+|N_{BS_n}(v_1)\setminus\{u,v,w\}|+|N_{BS_n}(v_2)\setminus\{u,v,w\}|=(2n-5)+(2n-5)+(2n-5)+(2n-6)+(2n-6)-3=10n-30$. It follows that $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\geq 1+10n-30=10n-29>4n-7$ $(n\geq 5)$, which contradicts $|F_2|\leq 4n-7$.

Suppose $F_1\setminus F_2=\emptyset$. Then $F_1\subseteq F_2$. Since F_2 is a nature faulty set, $BS_n-F_2=BS_n-F_1-F_2$ has no isolated vertex. The proof of Claim 1 is completed.

Let $u\in V(BS_n)\setminus(F_1\cup F_2)$. By Claim 1, u has at least one neighbor in $BS_n-F_1-F_2$. Since the vertex set pair (F_1,F_2) is not satisfied with any one condition in Theorem 5.3.2, by the condition (1) of Theorem 5.3.2, for any pair of adjacent vertices $u,w\in V(BS_n)\setminus(F_1\cup F_2)$, there is no vertex $v\in F_1\triangle F_2$ such that $uw\in E(BS_n)$ and $vw\in E(BS_n)$. It follows that u has no neighbor in $F_1\triangle F_2$. By the arbitrariness of u, there is no edge between $V(BS_n)\setminus(F_1\cup F_2)$ and $F_1\triangle F_2$. Since $F_2\setminus F_1\neq\emptyset$ and F_1 is a nature faulty set, $\delta_{BS_n}([F_2\setminus F_1])\geq 1$. By Lemma 8.2.1, $|F_2\setminus F_1|\geq 2$. Since both F_1 and F_2 are nature faulty sets, and there is no edge between $V(BS_n)\setminus(F_1\cup F_2)$ and $F_1\triangle F_2$, $F_1\cap F_2$ is a nature cut of BS_n. By Theorem 8.1.2, we have $|F_1\cap F_2|\geq 4n-8$. Therefore, $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\geq 2+(4n-8)=4n-6$, which contradicts $|F_2|\leq 4n-7$. Therefore, BS_n is nature $(4n-7)$-diagnosable and $t_1(BS_n)\geq 4n-7$. The proof is completed. □

Combining Lemmas 8.3.1 and 8.3.2, we have the following theorem.

Theorem 8.3.3 Let $n \geq 5$. Then the nature diagnosability of the bubble-sort star graph BS_n under MM* model is $4n-7$.

8.4 Conclusion

In this chapter, we investigated the problem of the nature diagnosability of Bubble-Sort Star Graph under the PMC model and MM* model. The transposition simple graph of Bubble-sort star graph combined both properties of the transposition simple graph of Bubble-sort graph and star graph. It makes the problem of investigating the nature diagnosability of Bubble-Sort Star Graph under the PMC model and MM* model more generalized and challenging.

Chapter 9

The Connectivity & Nature Diagnosability of Expanded k-Ary n-Cubes

In this chapter, we show that (1) the connectivity of XQ_n^k is $4n$; (2) the nature connectivity of XQ_n^k is $8n-4$; (3) the nature diagnosability of XQ_n^k under the PMC model and MM* model is $8n-3$ for $n \geq 2$. The results in this chapter is published in RAIRO - Theoretical Informatics and Applications [85].

9.1 Some Basic Propositions of Expanded k-Ary n-Cubes

We can partition XQ_n^k into k disjoint subgraphs $XQ_n^k[0]$, $XQ_n^k[1], \ldots, XQ_n^k[k-1]$ (abbreviated as $XQ[0]$, $XQ[1]$, $\ldots$, $XQ[k-1]$, if there is no ambiguity), where every vertex $u = u_0u_1\ldots u_{n-1} \in V(XQ_n^k)$ has a fixed integer i in the last position u_{n-1} for $i \in \{0,1,\ldots,k-1\}$. Let $u \in V(XQ[i])$. Then $N(u) \setminus V(XQ[i])$ is said to be outside neighbors of u.

Proposition 9.1.1 Each $XQ[i]$ is isomorphic to XQ_{n-1}^k for $0 \leq i \leq k-1$.

Proof: Note that the vertex set of XQ_{n-1}^k is $\{u_0u_1\ldots u_{n-2} : 0 \leq u_i \leq k-1, 0 \leq i \leq n-2\}$ and the vertex set of $XQ[i]$ is $\{u_0u_1\ldots u_{n-2}i : 0 \leq u_j \leq k-1, 0 \leq j \leq n-2, i \in \{0,1,\ldots,k-1\}\}$.

Therefore, $|\{u_0u_1\ldots u_{n-2} : 0 \le u_i \le k-1, 0 \le i \le n-2\}| = |\{u_0u_1\ldots u_{n-2}i : 0 \le u_j \le k-1, 0 \le j \le n-2, i \in \{0,1,\ldots,k-1\}\}|$. Now define a mapping from $V(XQ^k_{n-1})$ to $V(XQ[i])$ given by

$$\varphi: \;\; u_0u_1u_2\cdots u_{n-2} \to u_0u_1\cdots u_{n-2}i.$$

It is clear that φ is bijective. Let $u = u_0u_1u_2\cdots u_{n-2}$, $v = v_0v_1v_2\cdots v_{n-2}$, and $uv \in E(XQ^k_{n-1})$, then, based on the definition of XQ^k_{n-1}, there exists an integer $j \in \{0,1,\ldots,n-2\}$ such that $v_j = u_j + g$ (mod k) and $u_i = v_i$, for $i \in \{0,1,\ldots,n-2\}\setminus\{j\}$, where $g \in \{1,-1,2,-2\}$. Therefore, $\varphi(v) = v_0v_1v_2\cdots v_{n-2}i = u_0u_1\cdots u_{j-1}, u_j+g, u_{j+1}\cdots u_{n-2}i$. Note that $\varphi(u) = u_0u_1\cdots u_{j-1}, u_j, u_{j+1}\cdots u_{n-2}i$. Thus, $\varphi(u)\varphi(v) \in E(XQ[i])$.

Let $\varphi(u) = u_0u_1\cdots u_{j-1},\ u_j,\ u_{j+1}\cdots u_{n-2}i$, $\varphi(v) = v_0v_1v_2\cdots v_{n-2}i$ and $\varphi(u)\varphi(v) \in E(XQ[i])$, then there exists an integer $j \in \{0,1,\ldots,n-2\}$ such that $v_j = u_j + g$ (mod k) and $u_i = v_i$, for $i \in \{0,1,\ldots,n-2\}\setminus\{j\}$, where $g \in \{1,-1,2,-2\}$, i.e., $\varphi(v) = v_0v_1v_2\cdots v_{n-2}i = u_0u_1\cdots u_{j-1}, u_j+g, u_{j+1}\cdots u_{n-2}i$. Therefore, $\varphi^{-1}(v) = v_0v_1v_2\cdots v_{n-2} = u_0u_1\cdots u_{j-1}, u_j + g, u_{j+1}\cdots u_{n-2}$. Note that $\varphi^{-1}(u) = u_0u_1\cdots u_{j-1}, u_j, u_{j+1}\cdots u_{n-2}$. Thus, $uv = \varphi^{-1}(u)\varphi^{-1}(v) \in E(XQ^k_{n-1})$. □

Let $(Z_k)^n$ denotes the n-fold Cartesian product of the group $(Z_k, \oplus_k)$, where $Z_k = \{0,1,\ldots,k-1\}$ and where k denotes addition modulo k. Let $x = (x_0, x_1, \ldots, x_{n-1}) \in (Z_k)^n$. Then $x^{-1} = (k-x_0, k-x_1, \ldots, k-x_{n-1})$.

Here we will show that the expanded k-ary n-cube is Cayley graph.

Theorem 9.1.1 Let $n \ge 1$ and even $k \ge 6$. The expanded k-ary n-cube XQ^k_n is the Cayley graph $Cay(S,(Z_k)^n)$, where the spanning set S is $S = \{\pm e_1,\ldots,\pm e_n\} \cup \{\pm 2e_1,\ldots,\pm 2e_n\}$ with mod k.

Proof: Note that $V(XQ^k_n) = (Z_k)^n$. Now define a mapping from $V(XQ^k_n)$ to $(Z_k)^n$ given by

$$\varphi: \;\; u_1u_2u_3\cdots u_{n-1} \to u_1u_2\cdots u_{n-1}.$$

Then φ is bijective. Let $uv \in E(XQ^k_n)$. Then, the definition of XQ^k_n, there exists an integer $j \in \{0,1,\ldots,n-1\}$ such that $v_j = u_j + g$ (mod k) and $u_i = v_i$, for $i \in \{0,1,\ldots,n-1\}\setminus\{j\}$, where $g \in \{1,-1,2,-2\}$. Note that $k-1 \equiv -1 \pmod{k}$ and $k-2 \equiv -2 \pmod{k}$. Let $s = (0,\ldots,0,0+g,0,\ldots,0)$, and let $0+g$ be the j position in the s, then $s \in S$. Note that $\varphi(u)\varphi(v) = uv$. Therefore, $v = u+s$, hence $\varphi(u)\varphi(v) \in E(Cay(S,(Z_k)^n))$.

Let $\varphi(u)\varphi(v) \in E(Cay(S,(Z_k)^n))$, then by the definition of $Cay(S,(Z_k)^n)$, there exists an $s \in S$ such that $\varphi(v) = \varphi(u)+s$. Note that $\varphi(u) = u$ and $\varphi(v) = v$, therefore, $v = \varphi(v) = \varphi(u)+s = u+s$. Note that $\varphi^{-1}(u)\varphi^{-1}(v) = uv$ and $v = u+s$. Let $s = (0,\ldots,0,0+g,0,\ldots,0)$, and let $0+g$ be the j position in the s, then $v_j = u_j + g \pmod{k}$ and $u_i = v_i$, for $i \in \{0,1,\ldots,n-1\} \setminus \{j\}$. Note that $k-1 \equiv -1 \pmod{k}$ and $k-2 \equiv -2 \pmod{k}$, therefore, $g \in \{1,-1,2,-2\}$ and hence $uv \in E(XQ_n^k)$. □

By Theorem 9.1.1, we know that the expanded k-ary n-cube belongs to Cayley graph and hence XQ_n^k has the following properties since Cayley graphs are regular and vertex-transitive.

Proposition 9.1.2 XQ_n^k is $4n$-regular, vertex-transitive.

It is straightforward to see the following proposition.

Proposition 9.1.3 The girth of XQ_n^k is 3.

Combined with Proposition 9.1.3, Proposition 9.1.4 will play a significant role in proving the following lemmas and theorems throughout this chapter.

Proposition 9.1.4 Let $u \in V(XQ[i])$, then four outside neighbors of u are in four distinct $XQ[j]'$s.

Proof: Let $u = u_0u_1\ldots u_{n-2}i$, then $u \in V(XQ[i])$, $u_0u_1\ldots u_{n-2}i+1 \in V(XQ[i+1])$, $u_0u_1\ldots u_{n-2}i-1 \in V(XQ[i-1])$, $u_0u_1\ldots u_{n-2}i+2 \in V(XQ[i+2])$ and $u_0u_1\ldots u_{n-2}i-2 \in V(XQ[i-2])$. □

The following propositions show how large is the common neighbourhood of two vertices in expanded k-ary 1-cube and then in expanded k-ary n-cubes. These two propositions will be important parts in the proof to determine nature connectivity of expanded k-ary n-cubes.

Proposition 9.1.5 Let XQ_1^k be the expanded k-ary 1-cube.

(1) If $k=6$ and two vertices u,v are adjacent, then there are at most two common neighbors of these two vertices, i.e., $|N(u)\cap N(v)|\leq 2$. If $k=6$ and two vertices u,v are not adjacent, then there are at most four common neighbors of these two vertices, i.e., $|N(u)\cap N(v)|\leq 4$.

(2) If $k\geq 8$, then there are at most two common neighbors of two vertices u,v, i.e., $|N(u)\cap N(v)|\leq 2$.

Proof: Let $u,v\in V(XQ_1^k)$, suppose that $k=6$, then $XQ_1^k=XQ_1^6$. By Proposition 9.1.2, without loss of generality, we assume that $u=0$. Since $N(0)=\{1,2,4,5\}$ and $N(3)=\{1,2,4,5\}$, furthermore two vertices $0,3$ are not adjacent and $N(0)\cap N(3)=\{1,2,4,5\}$. Therefore, there are at most four common neighbors of these two vertices, i.e., $|N(u)\cap N(v)|\leq 4$. Fig. 2.9 (geometry) is symmetrical on the axis 03. Therefore, we consider only edges 01 and 02 for adjacent two vertices. Note that $N(0)=\{1,2,4,5\}$ and $N(1)=\{0,2,3,5\}$, thus $N(0)\cap N(1)=\{2,5\}$, $N(0)=\{1,2,4,5\}$ and $N(2)=\{0,1,3,4\}$. Therefore, $N(0)\cap N(2)=\{1,4\}$. So, for adjacent two vertices u,v, there are at most two common neighbors of these two vertices, i.e., $|N(u)\cap N(v)|\leq 2$.

Suppose that $k\geq 8$. By Proposition 9.1.2, we further suppose that $u=0$. Fig. 2.10 (geometry) is symmetrical about the axis $0\frac{k}{2}$. Therefore, we only consider two vertices: $u=0$ and $v\in\{1,2,\ldots,\frac{k}{2}\}$. Since $N(0)=\{1,2,k-2,k-1\}$, $N(1)=\{0,2,3,k-1\}$ and $N(2)=\{0,1,3,4\}$, so $N(0)\cap N(1)=\{2,k-1\}$ and $N(0)\cap N(2)=\{1\}$. Thus, for adjacent two vertices u,v, there are at most two common neighbors of these two vertices, i.e., $|N(u)\cap N(v)|\leq 2$. Now consider two vertices: $u=0$ and $v\in\{3,4,\ldots,\frac{k}{2}\}$. Let $v=3$. Note that $N(3)=\{1,2,4,5\}$, so $N(0)\cap N(3)=\{1,2\}$. Note that $N(4)=\{2,3,5,6\}$, therefore, $N(0)\cap N(4)=\{2,6\}$ when $k=8$ and $N(0)\cap N(4)=\{2\}$ when $k\geq 10$. Let $v\in\{5,6,\ldots,\frac{k}{2}\}$ and $x\in N(v)$, then $3\leq x\leq k-3$. So $N(0)\cap N(x)=\emptyset$. Thus, there are at most two common neighbors of these two vertices u,v, i.e., $|N(u)\cap N(v)|\leq 2$. $\square$

Proposition 9.1.6 Let XQ_n^k be the expanded k-ary n-cube.

(1) If $k=6$ and two vertices u,v are adjacent, then there are at most two common neighbors of these two vertices, i.e., $|N(u)\cap N(v)|\leq 2$. If $k=6$ and two vertices u,v are

not adjacent, then there are at most four common neighbors of these two vertices, i.e., $|N(u) \cap N(v)| \leq 4$.

(2) If $k \geq 8$, then there are at most two common neighbors of two vertices u, v, i.e., $|N(u) \cap N(v)| \leq 2$.

Proof: We can partition XQ_n^k into k disjoint subgraphs $XQ_n^k[0], XQ_n^k[1], \ldots, XQ_n^k[k-1]$ (abbreviated as $XQ[0], XQ[1], \ldots, XQ[k-1]$, if there is no ambiguity), where every vertex $u_0u_1 \ldots u_{n-1} \in V(XQ_n^k)$ has a fixed integer i in the last position u_{n-1} for $i \in \{0, 1, \ldots, k-1\}$. By Proposition 9.1.1, each $XQ[i]$ is isomorphic to XQ_{n-1}^k for $0 \leq i \leq k-1$. Let $u, v \in V(XQ_n^k)$, by Proposition 9.1.2, without loss of generality, we suppose that $u = \underbrace{00\ldots0}_{n}$, then $u \in V(XQ[0])$.

Suppose that $k = 6$. When $n = 1$, the result holds by Proposition 9.1.5. We proceed by induction on n ($n \geq 2$). Our induction hypothesis is the following.

(a) If two vertices u, v are adjacent, then there are at most two common neighbors of these two vertices, i.e., $|N(u) \cap N(v)| \leq 2$ in XQ_{n-1}^6.

(b) If two vertices u, v are not adjacent, then there are at most four common neighbors of these two vertices, i.e., $|N(u) \cap N(v)| \leq 4$ in XQ_{n-1}^6.

Let $v \in V(XQ[0])$, by the induction hypothesis, (a) if two vertices u, v are adjacent, $|N(u) \cap N(v)| \leq 2$ in $XQ[0]$; (b) if two vertices u, v are not adjacent, $|N(u) \cap N(v)| \leq 4$ in $XQ[0]$ and also by the Proposition 9.1.4, $(N(u) \cap V(XQ[i])) \cap (N(v) \cap V(XQ[i])) = \emptyset$ for $i \in \{1, 2, \ldots, 5\}$. Therefore, $|N(u) \cap N(v)| \leq 2$ for (a) and $|N(u) \cap N(v)| \leq 4$ for (b) in this case.

Suppose that $v \in V(XQ[i])$ for $i \in \{1, 2, \ldots, 5\}$. If $v \in \{\underbrace{0\ldots0}_{n-1}1, \underbrace{0\ldots0}_{n-1}2, \ldots, \underbrace{0\ldots0}_{n-1}4, \underbrace{0\ldots0}_{n-1}5\}$, then, by the induction hypothesis, (a) if two vertices u, v are adjacent, $|N(u) \cap N(v)| \leq 2$; (b) if two vertices u, v are not adjacent, $|N(u) \cap N(v)| \leq 4$. Note that $(N(u) \cap V(XQ[i])) \cap (N(v) \cap V(XQ[i])) \backslash \{\underbrace{0\ldots0}_{n-1}1, \underbrace{0\ldots0}_{n-1}2, \ldots, \underbrace{0\ldots0}_{n-1}4, \underbrace{0\ldots0}_{n-1}5\} = \emptyset$ for $i \in \{0, 1, 2, \ldots, 5\}$. Therefore, $|N(u) \cap N(v)| \leq 2$ or $|N(u) \cap N(v)| \leq 4$ in this case. Let $v \in V(XQ[i]) \backslash \{\underbrace{0\ldots0}_{n-1}1, \underbrace{0\ldots0}_{n-1}2, \underbrace{0\ldots0}_{n-1}3, \underbrace{0\ldots0}_{n-1}4, \underbrace{0\ldots0}_{n-1}5\}$ for $i \in \{1, 2, 3, 4, 5\}$. Since $|N(u) \cap V(XQ[i])| \leq 1$ for $i \in \{1, 2, 3, 4, 5\}$,

$|N(v) \cap V(XQ[0])| \leq 1$ and $(N(u) \cap V(XQ[j])) \cap (N(v) \cap V(XQ[j])) = \emptyset$ for $i \neq j$, $|N(u) \cap N(v)| \leq 2$ holds.

Suppose that $k \geq 8$. When $n = 1$, the result holds by Proposition 9.1.5. We proceed by induction on n. Our induction hypothesis is that $|N(u) \cap N(v)| \leq 2$ for two vertices u, v in XQ^k_{n-1}. Let $v \in V(XQ[0])$. By the induction hypothesis, $|N(u) \cap N(v)| \leq 2$ for two vertices u, v in $XQ[0]$. By Proposition 9.1.4, $(N(u) \cap V(XQ[i])) \cap (N(v) \cap V(XQ[i])) = \emptyset$ for $i \in \{1, 2, \ldots, k-1\}$. Therefore, $|N(u) \cap N(v)| \leq 2$ in this case.

Suppose that $v \in V(XQ[i])$ for $i \in \{1, 2, \ldots, k-2, k-1\}$. If $v \in \{\underbrace{0\ldots0}_{n-1}1, \underbrace{0\ldots0}_{n-1}2, \ldots, \underbrace{0\ldots0}_{n-1}(k-1)\}$, then $|N(u) \cap N(v)| \leq 2$ by Propositions 9.1.4 and 9.1.5. Let $v \in V(XQ[i]) \setminus \{\underbrace{0\ldots0}_{n-1}1, \underbrace{0\ldots0}_{n-1}2, \ldots, \underbrace{0\ldots0}_{n-1}(k-1)\}$. Note that $|N(u) \cap V(XQ[i])| \leq 1$, $|N(v) \cap V(XQ[0])| \leq 1$ and $(N(u) \cap V(XQ[j])) \cap (N(v) \cap V(XQ[j])) = \emptyset$ for $i \neq j$. Therefore, there are at most two common neighbors of two vertices u, v, i.e., $|N(u) \cap N(v)| \leq 2$. $\square$

9.2 The Connectivity of Expanded k-Ary n-Cubes

To investigate the nature diagnosability of the expanded k-ary n-cube XQ^k_n, we need to know the nature connectivity of XQ^k_n. In this section, we shall show the connectivity and nature connectivity of XQ^k_n.

Proposition 9.2.1 The connectivity $\kappa(XQ^k_1) = 4$.

Proof: By Menger's Theorem, a graph XQ^k_1 has connectivity $\kappa(XQ^k_1) = 4$ if and only if, given any two distinct vertices of $V(XQ^k_1)$, there are 4 vertex-disjoint paths joining them. By Theorem 9.1.1, it is sufficient to show that, for $u = 0$ and a distinct vertex v of $V(XQ^k_1)$, there are 4 vertex-disjoint paths joining u and v. By the symmetry, we will prove that, for $u = 0$ and one $v \in \{1, 2, \ldots, \frac{k}{2}\}$, there are 4 vertex-disjoint paths joining u and v. Let an odd $i \in \{2, 3, \ldots, \frac{k}{2}\}$. We have that four vertex-disjoint paths: $0, 1, 3, 5, \ldots, i$; $0, 2, 4, \ldots, i-1, i$; $0, k-1, k-3, k-5, \ldots, i$ and $0, k-2, k-4, \ldots, i+1, i$. When $i = 1$, we have that four vertex-disjoint paths: $0, 1$; $0, k-1, 1$; $0, 2, 1$ and $0, k-2, k-4, \ldots, 4, 3, 1$.

Let an even $i \in \{1,2,3,\dots,\frac{k}{2}\}$. We have that four vertex-disjoint paths: $0,1,3,\dots,i-1,i$; $0,2,4,\dots,i$; $0,k-1,k-3,k-5,\dots,i+1,i$ and $0,k-2,k-4,\dots,i$. $\square$

Proposition 9.2.2 The connectivity $\kappa(XQ_2^k)=8$.

Proof: Note $\kappa(XQ_2^k) \le \delta(XQ_2^k)=8$. We prove this statement by contradiction. Suppose that $F \subseteq V(XQ_2^k)$ with $|F|\le 7$ is a cut of XQ_2^k. By Proposition 9.1.1, each $XQ[i]$ is isomorphic to XQ_1^k for $0\le i\le k-1$. Let $F_i = F\cap V(XQ[i])$ for $i\in\{0,1,2,\dots,k-1\}$.

Suppose that $|F_i| = \max\{|F_i| : 0\le i\le k-1\}$. Note that the vertex set of $XQ[i]$ is $\{u_0 i : 0\le u_0\le k-1, i\in\{1,\dots,k-1\}\}$ and the vertex set of $XQ[0]$ is $\{u_0 0 : 0\le u_0\le k-1\}$. Now define a mapping from $V(XQ_2^k)$ to $V(XQ_2^k)$ given by

$$\varphi:\ \ u_0u_1 \to u_0(u_1-i).$$

Then $\varphi(u_0 i)=u_0 0$.

Claim 1. φ is an automorphism of XQ_2^k.

It is clear that φ is bijective. Let $u=u_0u_1$, $v=v_0v_1$, and $uv\in E(XQ_2^k)$. Then, the definition of XQ_2^k, $v_0=u_0+g \pmod k$ and $v_1=u_1$, or $v_0=u_0$, $v_1=u_1+g \pmod k$, where $g\in\{1,-1,2,-2\}$. Suppose, firstly, that $v_0=u_0+g \pmod k$ and $v_1=u_1$. Note $\varphi(u)=u_0,u_1-i$ and $\varphi(v)=\varphi(u_0+g,u_1)=u_0+g,u_1-i$. Suppose, secondly, that $v_0=u_0$, $v_1=u_1+g \pmod k$. Note $\varphi(u)=u_0,u_1-i$ and $\varphi(v)=\varphi(u_0,u_1+g)=u_0,u_1+g-i$. Therefore, $\varphi(u)\varphi(v)\in E(XQ_2^k)$ by the definition of XQ_2^k.

Let $\varphi(u)=u_0,u_1-i$, $\varphi(v)=v_0,v_1-i$ and $\varphi(u)\varphi(v)\in E(XQ_2^k)$, then, by the definition of XQ_2^k, $v_0=u_0+g \pmod k$ and $v_1-i=u_1-i$, or $v_0=u_0$, $v_1-i=u_1-i+g \pmod k$, where $g\in\{1,-1,2,-2\}$. Suppose, firstly, that $v_0=u_0+g \pmod k$ and $v_1-i=u_1-i$. Then $\varphi^{-1}(u)=u_0u_1$ and $\varphi^{-1}(v)=u_0+g,u_1$. Suppose, secondly, that $v_0=u_0$, $v_1-i=u_1-i+g \pmod k$. Then $\varphi^{-1}(u)=u_0u_1$ and $\varphi^{-1}(v)=u_0,u_1+g$. Therefore, $uv=\varphi^{-1}(u)\varphi^{-1}(v)\in E(XQ_{n-1}^k)$ by the definition of XQ_2^k. Thus, φ is an automorphism.

Claim 2. Let φ be defined as above. If $F\subseteq V(XQ_2^k)$ is a cut of XQ_2^k, then $\varphi(F)$ is also a cut of XQ_2^k. In particular, $\varphi(F_i)\subseteq V(XQ[0])$ and $|\varphi(F_i)|=|F_i|$.

Since φ is bijective, $|\varphi(F)| = |F|$ and $|\varphi(F_i)| = |F_i|$. Let $B_1, \ldots, B_k$ $(k \geq 2)$ be the components of $XQ_2^k - F$. Then $[V(B_i), V(B_j)] = \emptyset$ for $1 \leq i, j \leq k$ and $i \neq j$. Let $b_i \in V(B_i)$ and $b_j \in V(B_j)$. Then b_i is not adjacent to b_j. Since φ is an automorphism, $\varphi(b_i)$ is not adjacent to $\varphi(b_j)$. Therefore, $[\varphi(V(B_i)), \varphi(V(B_j))] = \emptyset$ for $1 \leq i, j \leq k$ and $i \neq j$, and hence $\varphi(F)$ is also a cut of XQ_2^k. Let $f \in F_i$, then $f = u_0 i$ for $0 \leq u_0 \leq k-1$. Therefore, $\varphi(f) = u_0 0 \in V(XQ[0])$ and hence $\varphi(F_i) \subseteq V(XQ[0])$.

By Claim 2, without loss of generality, we suppose that $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$. We consider the following cases.

Case 1. $|F_0| = 1$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, there are six F_i's such that $|F_i| = 1$ for $i \in \{1, 2, \ldots, k-1\}$ and $k \geq 8$. By Proposition 9.2.1, $XQ[i] - F_i$ is connected. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 2. $|F_0| = 2$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, there are at most five F_i's such that $1 \leq |F_i| \leq 2$ for $i \in \{1, 2, \ldots, k-1\}$. By Proposition 9.2.1, $XQ[i] - F_i$ is connected. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 3. $|F_0| = 3$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, there are at most four F_i's such that $1 \leq |F_i| \leq 3$ for $i \in \{1, 2, \ldots, k-1\}$. By Proposition 9.2.1, $XQ[i] - F_i$ is connected. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Without loss of generality, we suppose that $|F_1| = 3$. Then $|F_{k-1}| \leq 1$. Since there is a perfect matching between $XQ[0]$ and $XQ[k-1]$, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 4. $|F_0| = 4$.

In this case, there are at most three F_i's such that $1 \leq |F_i| \leq 3$ for $i \in \{1, 2, \ldots, k-1\}$. By Proposition 9.2.1, $XQ[i] - F_i$ is connected. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Since $|F_1|+|F_2|+\cdots+|F_{k-1}|=3$, by Proposition 9.1.4, XQ_2^k-F is connected, a contradiction to that F is a cut of XQ_2^k.

Case 5. $|F_0|=5$.

In this case, there are at most two F_i's such that $1\le|F_i|\le 2$ for $i\in\{1,2,\ldots,k-1\}$.By Proposition 9.2.1, $XQ[i]-F_i$ is connected. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i\in\{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[1]-F_1)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_1|+|F_2|+\cdots+|F_{k-1}|=2$, by Proposition 9.1.4, XQ_2^k-F is connected, a contradiction to that F is a cut of XQ_2^k.

Case 6. $|F_0|=6$.

In this case, there exists a F_i's such that $|F_i|=1$ where $i\in\{1,2,\ldots,k-1\}$. By Proposition 9.2.1, $XQ[i]-F_i$ is connected. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i\in\{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[1]-F_1)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_1|+|F_2|+\cdots+|F_{k-1}|=1$, by Proposition 9.1.4, XQ_2^k-F is connected, a contradiction to that F is a cut of XQ_2^k.

Case 7. $|F_0|=7$.

In this case, $|F_1|=|F_2|=\cdots=|F_{k-1}|=0$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i\in\{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[1]-F_1)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_1|+|F_2|+\cdots+|F_{k-1}|=0$, by Proposition 9.1.4, XQ_2^k-F is connected, a contradiction to that F is a cut of XQ_2^k.

By Cases 1-7, The connectivity XQ_2^k is 8. □

Theorem 9.2.1 Let XQ_n^k be the expanded k-ary n-cube with $n\ge 1$ and even $k\ge 6$, then the connectivity $\kappa(XQ_n^k)=4n$.

Proof: We can partition XQ_n^k into k disjoint subgraphs $XQ_n^k[0]$, $XQ_n^k[1],\ldots,XQ_n^k[k-1]$ (abbreviated as $XQ[0]$,$XQ[1]$, $\ldots$, $XQ[k-1]$, if there is no ambiguity), where every vertex $u=u_0u_1\ldots u_{n-1}\in V(XQ_n^k)$ has a fixed integer i in the last position u_{n-1} for $i\in\{0,1,\ldots,k-1\}$. When $n=1$ and $n=2$, the result holds by Propositions 9.2.1 and 9.2.2. We proceed by induction on n. Our induction hypothesis is $\kappa(XQ_{n-1}^k)=4n-4$ when $n\ge 3$. By Proposition 9.1.1, each $XQ[i]$ is isomorphic to XQ_{n-1}^k for $0\le i\le k-1$. We will prove $\kappa(XQ_n^k)=4n$.

Suppose that $F \subseteq V(XQ_n^k)$ is a minimum cut of XQ_n^k. Since $\kappa(XQ_n^k) \leq \delta(XQ_n^k) = 4n$, $|F| \leq 4n$ holds. It is sufficient to show that $XQ_n^k - F$ is connected for $|F| \leq 4n-1$. We prove this statement by contradiction. Suppose that $F \subseteq V(XQ_n^k)$ with $|F| \leq 4n-1$ is a cut of XQ_n^k. Let $F_i = F \cap V(XQ[i])$ for $i \in \{0,1,2,\ldots,k-1\}$ with $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$. We consider the following cases.

Case 1. $|F_0| \leq 4n-5$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4n-5$. By the induction hypothesis, $XQ[i] - F_i$ is connected. Since $k^{n-1} > 4n-5+(4n-5) = 8n-10$ and there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0,1,\ldots,k-2\}$, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 2. $4n-4 \leq |F_0| \leq 4n-1$.

In this case, there are at most three F_i's such that $1 \leq |F_i| \leq 3$. By Proposition 9.2.1, $XQ[i] - F_i$ is connected for $i \in \{1,2,\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0,1,\ldots,k-2\}$, $XQ_n^k[V(XQ[1]-F_1) \cup \cdots \cup V(XQ[k-1]-F_{k-1})]$ is connected. By Proposition 9.1.4, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

From Cases 1 and 2, The connectivity XQ_n^k is $4n$. □

Theorem 9.2.2 Let XQ_n^k be the expanded k-ary n-cube with $n \geq 1$ and even $k \geq 6$, then XQ_n^k is tightly $4n$ super-connected.

Proof: Let $F \subseteq V(XQ_n^k)$ with $|F| = 4n$ be any minimum cut of XQ_n^k, also let $F_i = F \cap V(XQ[i])$ for $i \in \{0,1,2,\ldots,k-1\}$ with $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, we consider the following cases.

Case 1. $|F_0| \leq 4n-5$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4n-5$, by Theorem 9.2.1, $XQ[i] - F_i$ is connected. Since $k^{n-1} > 4n-5+(4n-5) = 8n-10$ and there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0,1,\ldots,k-2\}$, then $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 2. $|F_0| = 4n-4$.

Suppose that there is only one F_i such that $|F_i| \neq 0$, we know that $|F_i| = 4$. Without loss of generality, we suppose that $|F_1| = 4$. By Proposition 9.2.1, $XQ[i] - F_i$ is connected for $i \in \{2, 3, \ldots, k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[2] - F_3) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Since $|F_{k-1}| = 0$ (or $|F_2| = 0$) and there is a perfect matching between $XQ[0]$ and $XQ[k-1]$ (or $XQ[0]$ and $XQ[2]$), $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Suppose that there are two F_i's such that $|F_i| \neq 0$, then we know $|F_i| \leq 3$. By Proposition 9.2.1, $XQ[i] - F_i$ is connected for $i \in \{1, 2, \ldots, k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, and $XQ_2^k[V(XQ[1] - F_1) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected. By Proposition 9.1.4, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Suppose that there are three F_i's such that $|F_i| \neq 0$, then $|F_i| \leq 2$. By Proposition 9.2.1, $XQ[i] - F_i$ is connected for $i \in \{1, 2, \ldots, k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, so $XQ_2^k[V(XQ[1] - F_1) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected. By Proposition 9.1.4, we have $XQ_n^k - F$ is connected, which is a contradiction to that F is a cut of XQ_n^k.

Suppose that there are four F_i's such that $|F_i| \neq 0$, then we have $|F_i| \leq 1$. By Proposition 9.2.1, $XQ[i] - F_i$ is connected for $i \in \{1, 2, \ldots, k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, so $XQ_2^k[V(XQ[1] - F_1) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Let $XQ[0] - F_0$ be connected. Since $k^{n-1} > 4n - 4 + 1 = 4n - 3$ and there is a perfect matching between $XQ[0]$ and $XQ[1]$, we know $XQ_n^k - F$ is connected, which is a contradiction to that F is a cut of XQ_n^k.

Let $XQ[0] - F_0$ be disconnected and let $B_1, \ldots, B_k$ $(k \geq 2)$ be the components of $XQ[0] - F_0$. If $k \geq 3$, then, by Proposition 9.1.4, $|(N(V(B_1) \cup N(V(B_2))) \cap (V(XQ[1] - F_1) \cup \cdots \cup V(XQ[k-1] - F_{k-1}))| \geq 8$. If $|V(B_r)| \geq 2$ $(1 \leq r \leq k-1)$, then, by Proposition 9.1.4, $|N(V(B_1) \cap (V(XQ[1] - F_1) \cup \cdots \cup (V(XQ[k-1] - F_{k-1}))| \geq 8$. Combining this with $|F_1| + \cdots + |F_{k-1}| = 4$, we have that $XQ[0] - F_0$ has two components, one of which is an isolated vertex v. Since $k^{n-1} > 4n - 4 + 1 + 1 = 4n - 2$ and there is a perfect matching between

$XQ[0]$ and $XQ[1]$, $XQ_n^k[V(XQ[0]-F_0-v)\cup V(XQ[1]-F_1)]\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected. Therefore, XQ_n^k-F has two components, one of which is an isolated vertex.

Case 3. $4n-3\leq|F_0|\leq 4n$.

In this case, there are at most three F_i's such that $1\leq|F_i|\leq 3$. By Proposition 9.2.1, $XQ[i]-F_i$ is connected for $i\in\{1,2,\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i\in\{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[1]-F_1)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected. By Proposition 9.1.4, XQ_n^k-F is connected, a contradiction to that F is a cut of XQ_n^k.

From Cases 1-3, XQ_n^k is tightly $4n$ super-connected. □

Here we give a proposition when $n=2$ to facilitate the understanding of Proposition 9.2.4.

Proposition 9.2.3 Let XQ_2^k be the expanded k-ary 2-cube with even $k\geq 6$, and let $F\subseteq V(XQ_2^k)$ with $|F|\leq 11$. If XQ_2^k-F is disconnected, then XQ_2^k-F has two components, one of which is an isolated vertex.

Proof: We can partition XQ_2^k into k disjoint subgraphs $XQ_2^k[0]$, $XQ_2^k[1],\ldots,XQ_2^k[k-1]$ (abbreviated as $XQ[0]$, $XQ[1]$, $\ldots$, $XQ[k-1]$, if there is no ambiguity), where every vertex $u_0u_1\in V(XQ_2^k)$ has a fixed integer i in the last position u_1 for $i\in\{0,1,\ldots,k-1\}$. By Proposition 9.1.1, each $XQ[i]$ is isomorphic to XQ_1^k for $0\leq i\leq k-1$. By Theorem 9.2.1, $\kappa(XQ[i])=4$. Let $F_i=F\cap V(XQ[i])$ for $i\in\{0,1,2,\ldots,k-1\}$ with $|F_0|=\max\{|F_i|:0\leq i\leq k-1\}$. We consider the following cases.

Case 1. $|F_0|\leq 3$.

Since $|F_0|=\max\{|F_i|:0\leq i\leq k-1\}$, $|F_i|\leq 3$. By Theorem 9.2.1, $XQ[i]-F$ is connected.

Suppose that $|F_0|\leq 2$. Then $|F_i|\leq 2$ for $i\in\{1,2,\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i\in\{0,1,\ldots,k-2\}$, XQ_2^k-F is connected, a contradiction to that F is a cut of XQ_2^k.

Suppose that $|F_0|=3$. Then $|F_i|\leq 3$ for $i\in\{1,2,\ldots,k-1\}$. If $|F_i|\leq 2$ for $i\in\{1,2,\ldots,k-1\}$, then XQ_2^k-F is connected, a contradiction to that F is a cut of XQ_2^k.

If $k \geq 8$, then $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k. Therefore, let $k = 6$ and there be F_i's for $i \in \{1,2,3,4,5\}$ such that $|F_i| = 3$. Since $|F_1| + \cdots + |F_5| \leq 8$, there are at most two F_i's such that $|F_i| = 3$. Suppose that there is one F_i such that $|F_i| = 3$. Without loss of generality, let that $|F_1| = 3$. Then $|F_5| \leq 2$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0,1,\ldots,4\}$, $Q_2^6[V(XQ[1]-F_1) \cup \cdots \cup V(XQ[5]-F_5)]$ is connected. Since there is a perfect matching between $XQ[0]$ and $XQ[5]$, $Q_2^6 - F$ is connected, a contradiction to that F is a cut of Q_2^6. Suppose that there are two F_i such that $|F_i| = 3$. Without loss of generality, let that $|F_1| = 3$ and $|F_5| = 3$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0,1,\ldots,4\}$, $Q_2^6[V(XQ[1]-F_1) \cup \cdots \cup V(XQ[5]-F_5)]$ is connected. Since there is a perfect matching between $XQ[0]$ and $XQ[2]$, $Q_2^6 - F$ is connected, a contradiction to that F is a cut of Q_2^6.

Case 2. $|F_0| = 4$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4$. Since $|F_1| + \cdots + |F_5| \leq 7$, there is at most one F_i such that $|F_i| = 4$ for $i \in \{1,2\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4$. Then $|F_2| + \cdots + |F_{k-1}| \leq 3$. By Theorem 9.2.1, $XQ[i] - F$ is connected for $i \in \{2,3,\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0,1,\ldots,4\}$, $XQ_2^k[V(XQ[2]-F_2) \cup \cdots \cup V(XQ[k-1]-F_{k-1})]$ is connected. By theorem 9.2.2, $XQ[i] - F_i$ is connected or $XQ[i] - F_i$ has two components, one of which is an isolated vertex v_i for $i \in \{0,1\}$. Let $XQ[i] - F_i$ be connected for $i \in \{1,2\}$. Then $|V(XQ[i]-F_i)| \geq 2$ for $i \in \{1,2\}$. By Proposition 9.1.4, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k. Without loss of generality, suppose that $XQ[1] - F_1$ has two components, one of which is an isolated vertex and $XQ[0] - F_0$ is connected. Since $|V(XQ[0]-F_0)| \geq 2$ and $|F_2| + \cdots + |F_{k-1}| \leq 3$, by Proposition 9.1.4, $XQ_2^k[V(XQ[0]-F_0) \cup V(XQ[2]-F_2) \cup \cdots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Therefore, $XQ_2^k - F$ is connected, or $XQ_2^k - F$ has two components, one of which is an isolated vertex. Then $XQ[i] - F_i$ is disconnected for $i \in \{1,2\}$. Suppose that $k = 6$. Then $XQ[i] - F_i$ has two components, two of which are isolated vertices for $i \in \{1,2\}$. Since $|F_2| + \cdots + |F_5| \leq 3$, by theorem 9.2.2, $XQ_2^6[V(XQ[i]-F_i) \cup V(XQ[2]-F_2) \cup \cdots \cup V(XQ[5]-F_5)]$ is connected, or $XQ_2^6[V(XQ[i]-F_i) \cup V(XQ[2]-F_2) \cup \cdots \cup V(XQ[5]-F_5)]$ has two components, one of which is an isolated vertex v_i for

$i \in \{0,1\}$. Note that $|N(v_0) \cap N(v_1)| \leq 2$. Since $|N(v_0) \cap N(v_1)| \leq 2$ and $|F_2| + \cdots + |F_5| \leq 3$, $XQ_2^6 - F$ is connected, or $XQ_2^6 - F$ has two components, one of which is an isolated vertex. Suppose that $k \geq 8$. Since $|V(XQ[0] - F_0)| \geq 3$ and $|F_2| + \cdots + |F_{k-1}| \leq 3$, $XQ_2^k[V(XQ[0] - F_0) \cup V(XQ[2] - F_2) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected, or $XQ_2^k[V(XQ[0] - F_0) \cup V(XQ[2] - F_2) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ has two components, one of which is an isolated vertex. If $XQ_2^k[V(XQ[0] - F_0) \cup V(XQ[2] - F_2) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected, then $XQ_2^k - F$ is connected, or $XQ_n^k - F$ has two components, one of which is an isolated vertex. Then $XQ_2^k[V(XQ[0] - F_0) \cup V(XQ[2] - F_2) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ has two components, one of which is an isolated vertex. Since $|V(XQ[1] - F_1)| \geq 3$, $XQ_2^k[V(XQ[1] - F_1) \cup V(XQ[2] - F_2) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected, or $XQ_2^k[V(XQ[1] - F_1) \cup V(XQ[2] - F_2) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ has two components, one of which is an isolated vertex. Suppose that $XQ_2^k[V(XQ[i] - F_i) \cup V(XQ[2] - F_2) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ has two components, one of which is an isolated vertex v_i for $i \in \{0,1\}$. By Proposition 9.1.6, $|N(v_0) \cap N(v_1)| \leq 2$. Since $|N(v_0) \cap N(v_1)| \leq 2$ and $|F_2| + \cdots + |F_{k-1}| \leq 3$, $XQ_2^k - F$ is connected, or $XQ_2^k - F$ has two components, one of which is an isolated vertex.

Suppose that there are at most three F_i's such that $|F_i| \neq 0$. Then $|F_i| \leq 3$ for $i \in \{2,3,\ldots,k-1\}$. By Theorem 9.2.1, $XQ[i] - F$ is connected for $i \in \{2,3,\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, XQ_2^k $[V(XQ[2] - F_2) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected. By Proposition 9.1.4, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 3. $|F_0| = 5$.

In this case, $|F_1| + \cdots + |F_{k-1}| \leq 11 - 5 = 6$. Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 5$ for $i \in \{1,2,\ldots,k-1\}$. Suppose that $|F_i| \leq 3$ for $i \in \{1,2,\ldots,k-1\}$. By Theorem 9.2.1, $XQ[i] - F$ is connected for $i \in \{1,2\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ (or $XQ[i]$ and $XQ[i+2]$), for $i \in \{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Since $|F_2| + \cdots + |F_5| \leq 6$, by Proposition 9.1.4, $XQ_2^k - F$ is connected, or $XQ_2^k - F$ has two components, one of which is an isolated vertex.

Note that there is at most one F_i such that $|F_i| = 4$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4$. Since $|F_1| + \cdots + |F_{k-1}| \leq 6$, there are at most three F_i's such that

$|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 9.1.4, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Note that there is at most one F_i such that $|F_i| = 5$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 5$. Since $|F_1| + \cdots + |F_{k-1}| \leq 6$, there are at most two F_i's such that $|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 9.1.4, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 4. $|F_0| = 6$.

In this case, $|F_1| + \cdots + |F_{k-1}| \leq 11 - 6 = 5$. Suppose that $|F_i| \leq 3$ for $i \in \{1,2,\ldots,k-1\}$. By Theorem 9.2.1, $XQ[i] - F$ is connected for $i \in \{1,2\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Since $|F_2| + \cdots + |F_5| \leq 5$, by Proposition 9.1.4, $XQ_2^k - F$ is connected, or $XQ_2^k - F$ has two components, one of which is an isolated vertex.

Note that there is at most one F_i such that $|F_i| = 4$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4$. Since $|F_1| + \cdots + |F_{k-1}| \leq 5$, there are at most two F_i's such that $|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 9.1.4, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Note that there is at most one F_i such that $|F_i| = 5$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 5$. Since $|F_1| + \cdots + |F_{k-1}| \leq 5$, there are at most one F_i such that $|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 9.1.4, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 5. $|F_0| = 7$.

In this case, $k \geq 8$ and $|F_1| + \cdots + |F_5| \leq 4$. Suppose that $|F_i| \leq 3$ for $i \in \{1,2,\ldots,k-1\}$. By Theorem 9.2.1, $XQ[i] - F$ is connected for $i \in \{1,2\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \cdots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Since $|F_2| + \cdots + |F_5| \leq 4$, by Proposition 9.1.4, $XQ_2^k - F$ is connected, or $XQ_2^k - F$ has two components, one of which is an isolated vertex.

Note that there is at most one F_i such that $|F_i| = 4$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4$. Since $|F_1| + \cdots + |F_{k-1}| \leq 4$, there are at most one F_i such that

$|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 9.1.4, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 6. $8 \leq |F_0| \leq 11$.

In this case, $|F_1| + \cdots + |F_5| \leq 3$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0,1,\ldots,k-2\}$, $Q_2^k[V(XQ[1]-F_1) \cup \cdots \cup V(XQ[k-1]-F_{k-1})]$ is connected. By Proposition 9.1.4, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k. $\square$

Then we show an important proposition for proving the nature connectivity of the expanded k-ary n-cube.

Proposition 9.2.4 Let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$, and let $F \subseteq V(XQ_n^k)$ with $|F| \leq 8n-5$. If $XQ_n^k - F$ is disconnected, then $XQ_n^k - F$ has two components, one of which is an isolated vertex.

Proof: We can partition XQ_n^k into k disjoint subgraphs $XQ_n^k[0]$, $XQ_n^k[1], \ldots, XQ_n^k[k-1]$ (abbreviated as $XQ[0]$, $XQ[1]$, $\ldots$, $XQ[k-1]$, if there is no ambiguity), where every vertex $u_0u_1\ldots u_{n-1} \in V(XQ_n^k)$ has a fixed integer i in the last position u_{n-1} for $i \in \{0,1,\ldots,k-1\}$. By Proposition 9.1.1, each $XQ[i]$ is isomorphic to XQ_{n-1}^k for $0 \leq i \leq k-1$. Let $F \subseteq V(XQ_n^k)$ with $|F| \leq 8n-5$ and let $XQ_n^k - F$ is disconnected. Let $F_i = F \cap V(XQ[i])$ for $i \in \{0,1,2,\ldots,k-1\}$ with $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$. When $n=2$, the result holds by Propositions 9.2.3. We proceed by induction on n. Our induction hypothesis is that $XQ_{n-1}^k - F$ has two components, one of which is an isolated vertex for $|F| \leq 8n-13$ and $n \geq 3$ if $XQ_{n-1}^k - F$ is disconnected. By Proposition 9.1.1, each $XQ[i]$ is isomorphic to XQ_{n-1}^k for $0 \leq i \leq k-1$. We consider the following cases.

Case 1. $|F_0| \leq 4n-5$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4n-5$ for $i \in \{1,2,\ldots,k-1\}$. By Theorem 9.2.1, $XQ[i] - F$ is connected for $i \in \{0,1,\ldots,k-1\}$. Since $k^{n-1} > 4n-5+(4n-5) = 8n-10$ and there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 2. $|F_0| = 4n-4$.

In this case, $|F_1|+\cdots+|F_{k-1}| \leq 8n-5-(4n-4) = 4n-1$. Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4n-4$ for $i \in \{1,2,\ldots,k-1\}$. Therefore, there is at most one F_i such that $|F_i| = 4n-4$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4n-4$.

Suppose that there are four F_i's such that $|F_i| \neq 0$. Then $|F_i| \leq 1$ for $i \in \{2,3,\ldots,k-1\}$. By Theorem 9.2.1, $XQ[i]-F$ is connected for $i \in \{2,3,\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_n^k[V(XQ[2]-F_2)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected. By theorem 9.2.2, $XQ[i]-F_i$ is connected or $XQ[i]-F_i$ has two components, one of which is an isolated vertex v_i for $i \in \{0,1\}$. Let $XQ[i]-F_i$ be connected for $i \in \{1,2\}$. Note that $k^{n-1}-(4n-4) > 2$ and hence $|V(XQ[i]-F_i)| \geq 2$. By Proposition 9.1.4, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k. Without loss of generality, suppose that $XQ[1]-F_1$ has two components, one of which is an isolated vertex and $XQ[0]-F_0$ is connected. Since $|V(XQ[0]-F_0)| \geq 2$ and $|F_2|+\cdots+|F_{k-1}| = 3$, by Proposition 9.1.4, $XQ_n^k[V(XQ[0]-F_0)\cup V(XQ[2]-F_2)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected. Therefore, $XQ_n^k - F$ is connected, or $XQ_n^k - F$ has two components, one of which is an isolated vertex. Then $XQ[i]-F_i$ be disconnected for $i \in \{1,2\}$. Since $|V(XQ[0]-F_0)| \geq 3$ and $|F_2|+\cdots+|F_{k-1}| = 3$, $XQ_n^k[V(XQ[0]-F_0)\cup V(XQ[2]-F_2)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected, or $XQ_n^k[V(XQ[0]-F_0)\cup V(XQ[2]-F_2)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ has two components, one of which is an isolated vertex. If $XQ_n^k[V(XQ[0]-F_0)\cup V(XQ[2]-F_2)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected, then $XQ_n^k - F$ is connected, or $XQ_n^k - F$ has two components, one of which is an isolated vertex. Then $XQ_n^k[V(XQ[0]-F_0)\cup V(XQ[2]-F_2)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ has two components, one of which is an isolated vertex v_0. Since $|V(XQ[1]-F_1)| \geq 3$, $XQ_n^k[V(XQ[1]-F_1)\cup V(XQ[2]-F_2)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected, or $XQ_n^k[V(XQ[1]-F_1)\cup V(XQ[2]-F_2)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ has two components, one of which is an isolated vertex. Suppose that $XQ_n^k[V(XQ[i]-F_i)\cup V(XQ[2]-F_2)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ has two components, one of which is an isolated vertex v_i for $i \in \{0,1\}$. By Proposition 9.1.6,$|N(v_0)\cap N(v_1)| \leq 2$. Since $|N(v_0)\cap N(v_1)| \leq 2$ and $|F_2|+\cdots+|F_{k-1}| \leq 3$, $XQ_n^k - F$ is connected, or $XQ_n^k - F$ has two components, one of which is an isolated vertex.

Suppose that there are three F_i's such that $|F_i| \neq 0$. Then $|F_i| \leq 2$ for $i \in \{2,3,\ldots,k-1\}$. By Theorem 9.2.1, $XQ[i]-F$ is connected for $i \in \{2,3,\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_n^k[V(XQ[2]-F_2) \cup \cdots \cup V(XQ[k-1]-F_{k-1})]$ is connected. By Proposition 9.1.4, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 3. $|F_0| = 4n-3$.

In this case, $|F_1| + \cdots + |F_{k-1}| \leq 8n-5-(4n-3) = 4n-2$. Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4n-3$ for $i \in \{1,2,\ldots,k-1\}$. Suppose that $|F_i| \leq 4n-5$ for $i \in \{1,2,\ldots,k-1\}$. By Theorem 9.2.1, $XQ[i]-F$ is connected for $i \in \{1,2\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_n^k[V(XQ[1]-F_1) \cup \cdots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_0| = 4n-3 \leq 8n-13$, $XQ[0]-F_0$ has two components, one of which is an isolated vertex v_0 by the induction hypothesis. Since $k^{n-1} > 4n-3+4n-4+1 = 8n-6$, $XQ_n^k - F$ is connected, or has two components, one of which is an isolated.

Note that there is at most one F_i such that $|F_i| = 4n-4$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4n-4$. Since $|F_1| + \cdots + |F_{k-1}| \leq 4n-2$, there are three F_i's such that $|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 9.1.4, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Note that there is at most one F_i such that $|F_i| = 4n-3$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4n-3$. Since $|F_1| + \cdots + |F_{k-1}| \leq 4n-2$, there are two F_i's such that $|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 9.1.4, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 4. $|F_0| = 4n-2$.

In this case, $|F_1| + \cdots + |F_{k-1}| \leq 8n-5-(4n-2) = 4n-3$. Suppose that $|F_i| \leq 4n-5$ for $i \in \{1,2,\ldots,k-1\}$. By Theorem 9.2.1, $XQ[i]-F$ is connected for $i \in \{1,2\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_n^k[V(XQ[1]-F_1) \cup \cdots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_0| = 4n-2 \leq 8n-13$, $XQ[0]-F_0$ has two components, one of which is an isolated vertex v_0 by the induction hypothesis. Since $k^{n-1} > 4n-2+4n-4+1 = 8n-5$, $XQ_n^k - F$ is connected, or has two components, one of which is an isolated vertex. Note that there is at most one F_i such that $|F_i| = 4n-4$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4n-4$. Since $|F_2|+\cdots+|F_{k-1}| \leq 1$, By Proposition 9.1.4, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 5. $|F_0| = 4n-1$.

In this case, $|F_1|+\cdots+|F_{k-1}| \leq 8n-5-(4n-1) = 4n-4$. Suppose that $|F_i| \leq 4n-5$ for $i \in \{1,2,\ldots,k-1\}$. By Theorem 9.2.1, $XQ[i]-F$ is connected for $i \in \{1,2\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_n^k[V(XQ[1]-F_1)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_0| = 4n-1 \leq 8n-13$, $XQ[0]-F_0$ has two components, one of which is an isolated vertex v_0 by the induction hypothesis. Since $k^{n-1} > 4n-1+4n-4+1 = 8n-4$, $XQ_n^k - F$ is connected, or has two components, one of which is an isolated vertex. Note that there is at most one F_i such that $|F_i| = 4n-4$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4n-4$. Since $|F_2|+\cdots+|F_{k-1}| = 0$, By Proposition 9.1.4, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 6. $4n \leq |F_0| \leq 8n-13$.

In this case, $|F_1|+\cdots+|F_{k-1}| \leq 8n-5-4n = 4n-5$. By Theorem 9.2.1, $XQ[i]-F$ is connected for $i \in \{1,2,\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_n^k[V(XQ[1]-F_1)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected. Suppose that $XQ[0]$ is connected. Since $k^{n-1} > 8n-5$, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k. Then $XQ[0]$ is disconnected. By the induction hypothesis, $XQ[0]-F_0$ has two components, one of which is an isolated vertex. Since $k^{n-1} > 8n-5+1 = 8n-4$, $XQ_n^k - F$ is connected, or has two components, one of which is an isolated vertex.

Case 7. $8n-12 \leq |F_0| \leq 8n-5$.

In this case, $|F_1|+\cdots+|F_{k-1}| \leq 7$. Since $n \geq 3$, $\kappa(XQ[i]) = 4(n-1) \geq 8$ holds for $i \in \{1,2,\ldots,k-1\}$ by Theorem 9.2.1. By Theorem 9.2.1, $XQ[i]-F_i$ is connected for $i \in \{1,2,\ldots,k-1\}$. Since there is a perfect matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_n^k[V(XQ[1]-F_1)\cup\cdots\cup V(XQ[k-1]-F_{k-1})]$ is connected. Suppose that $XQ[0]-F_0$ is connected. Since $k^{n-1}>8n-5$ and there is a perfect matching between $XQ[0]$ and $XQ[1]$, XQ_n^k-F is connected, a contradiction to that F is a cut of XQ_n^k. Then $XQ[0]-F_0$ is disconnected. Let $B_1,\ldots,B_k$ $(k\geq 2)$ be the components of $XQ[0]-F_0$. If $k\geq 3$, then, by Proposition 9.1.4, $|(N(V(B_1)\cup V(B_2))\cap(V(XQ[1])\cup\cdots\cup V(XQ[k-1]))|\geq 8$. If $|V(B_j)|\geq 2$, then, by Proposition 9.1.4, $|N(V(B_j))\cap(V(XQ[1])\cup\cdots\cup V(XQ[k-1]))|\geq 8$ $(1\leq j\leq k)$. Combining this with $|F_1|+\cdots+|F_{k-1}|\leq 7$, we have that XQ_n^k-F is connected or XQ_n^k-F has two components, one of which is an isolated vertex. □

Lemma 9.2.3 Let $A=\{\underbrace{0\ldots0}_{n},1\underbrace{0\ldots0}_{n-1}\}$. If $F_1=N_{XQ_n^k}(A)$, $F_2=A\cup N_{XQ_n^k}(A)$, then $|F_1|=8n-4$, $|F_2|=8n-2$, $\delta(XQ_n^k-F_1)\geq 1$, and $\delta(XQ_n^k-F_2)\geq 1$ ($n\geq 2$ or $n=1$ and $k\geq 8$)(See Fig. 5.3).

Proof: By $A=\{\underbrace{0\ldots0}_{n},1\underbrace{0\ldots0}_{n-1}\}$, we have $XQ_n^k[A]=K_2$. From calculating, we have $|F_1|=|N_{XQ_n^k}(A)|=8n-4$ and $|F_2|=|A|+|F_1|=8n-2$ by Proposition 9.1.3. Suppose $n=1$ and $k\geq 8$. From Fig. 2.10, $XQ_1^k-F_2$ is connected. Therefore, $\delta(XQ_1^k-F_1)\geq 1$ and $\delta(XQ_1^k-F_2)\geq 1$. Let $n\geq 2$, $k\geq 8$ and $x\in V(XQ_n^k)\setminus F_2$. By Proposition 9.1.6, $|N_{XQ_n^k}(x)\cap F_2)|\leq 4$. Therefore, $\delta(XQ_n^k-F_2)\geq 4n-4\geq 1$. Let $n\geq 3$, $k=6$ and $x\in V(XQ_n^k)\setminus F_2$. By Proposition 9.1.6, $|N_{XQ_n^k}(x)\cap F_2)|\leq 8$. Therefore, $\delta(XQ_n^k-F_2)\geq 4n-8\geq 1$.

Let $n=2$, $k=6$ and $x\in V(XQ_2^6)\setminus F_2$. Then $V(XQ[0])-F_2=\emptyset$. Suppose that $x\in V(XQ[i])\setminus F_2$ for $i\in\{1,2,\ldots,5\}$. Let $u=00$ and $v=10$. If $x\in\{01,02,03,04,05\}$, then $x=03$.

Note $|N(x)\cap N(v)|=0$ and hence $|N_{XQ_2^6}(x)\cap F_2)|\leq 4$ in this case. Let $x\in V(XQ[i])\setminus\{01,02,03,04,05\}$ for $i\in\{1,2,3,4,5\}$. Since $|N(u)\cap V(XQ[i])|\leq 1$ for $i\in\{1,2,3,4,5\}$, $|N(x)\cap V(XQ[0])|\leq 1$, $|N(u)\cap N(x)|\leq 2$ holds. Similarly, $|N(v)\cap N(x)|\leq 2$. Therefore, $|N_{XQ_2^6}(x)\cap F_2)|\leq 4$ and hence $\delta(XQ_2^6-F_2)\geq 4\times 2-4\geq 1$. Note that $XQ_2^6-F_1$ has two parts $XQ_2^6-F_2$ and $XQ_2^6[A]=K_2$. Note that $\delta(XQ_2^6[A])=1$. Therefore, $\delta(XQ_2^6-F_1)\geq 1$. □

Theorem 9.2.4 Let XQ_n^k be the expanded k-ary n-cube with $n\geq 1$ and even $k\geq 6$, Then the nature connectivity of XQ_n^k is $8n-4$, i.e., $\kappa^*(XQ_n^k)=8n-4$.

Proof: Let $A = \{\underbrace{0\ldots0}_{n}, 1\underbrace{0\ldots0}_{n-1}\}$ in Lemma 9.2.3. Then $|N(A)| = 8n-4$. Since $N(A)$ is a nature cut of XQ_n^k, $\kappa^*(XQ_n^k) \leq 8n-4$ holds.

By Proposition 9.2.4, if $F \subseteq V(XQ_n^k)$ with $|F| \leq 8n-5$, then $XQ_n^k - F$ is connected or $XQ_n^k - F$ has two components, one of which is an isolated vertex. Therefore, if F is a nature cut of XQ_n^k, then $|F| \geq 8n-4$. Combining this with $\kappa^*(XQ_n^k) \leq 8n-4$, we have that $\kappa^*(XQ_n^k) = 8n-4$. $\square$

9.3 The Nature Diagnosability of Expanded k-Ary n-Cubes under the PMC Model

In this section, we shall show the nature diagnosability of the expanded k-ary n-cube under the PMC model.

Firstly we give the lower bound of the nature diagnosability of the expanded k-ary n-cube under PMC model with even $k \geq 6$.

Lemma 9.3.1 Let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$. Then the nature diagnosability of XQ_n^k under the PMC model is less than or equal to $8n-3$, i.e., $t_1(XQ_n^k) \leq 8n-3$.

Proof: Let A be defined in Lemma 9.2.3, and let $F_1 = N_{XQ_n^k}(A)$, $F_2 = A \cup N_{XQ_n^k}(A)$. By Lemma 9.2.3, $|F_1| = 8n-4$, $|F_2| = 8n-2$, $\delta(XQ_n^k - F_1) \geq 1$ and $\delta(XQ_n^k - F_2) \geq 1$. Therefore, F_1 and F_2 are both nature faulty sets of XQ_n^k with $|F_1| = 8n-4$ and $|F_2| = 8n-2$. Since $A = F_1 \triangle F_2$ and $N_{XQ_n^k}(A) = F_1 \subset F_2$, there is no edge of XQ_n^k between $V(XQ_n^k) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 5.2.2, we can deduce that XQ_n^k is not nature $(8n-2)$-diagnosable under the PMC model. Hence, by the definition of the nature diagnosability, we conclude that the nature diagnosability of XQ_n^k is less than $8n-2$, i.e., $t_1(XQ_n^k) \leq 8n-3$. $\square$

Secondly we prove the upper bound of the nature diagnosability of the expanded k-ary n-cube under PMC model with even $k \geq 6$.

Lemma 9.3.2 Let $n \geq 2$ and let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$. Then the nature diagnosability of XQ_n^k under the PMC model is more than or equal to $8n-3$, i.e., $t_1(XQ_n^k) \geq 8n-3$.

Proof: By the definition of the nature diagnosability, it is sufficient to show that XQ_n^k is nature $(8n-3)$-diagnosable. By Theorem 5.2.2, to prove XQ_n^k is nature $(8n-3)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(XQ_n^k)$ with $u \in V(XQ_n^k) \backslash (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of nature faulty subsets F_1 and F_2 of $V(XQ_n^k)$ with $|F_1| \leq 8n-3$ and $|F_2| \leq 8n-3$.

We prove this statement by contradiction. Suppose that there are two distinct nature faulty subsets F_1 and F_2 of $V(XQ_n^k)$ with $|F_1| \leq 8n-3$ and $|F_2| \leq 8n-3$, but the vertex set pair (F_1, F_2) does not satisfy the condition in Theorem 5.2.2, i.e., there are no edges between $V(XQ_n^k) \backslash (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Suppose $V(XQ_n^k) = F_1 \cup F_2$. By the definition of XQ_n^k, $|F_1 \cup F_2| = k^n$. It is obvious that $k^n > 16n-6$ for $n \geq 2$. Since $n \geq 5$, we have that $k^n = |V(XQ_n^k)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(8n-3) = 16n-6$, a contradiction. Therefore, $V(XQ_n^k) \neq F_1 \cup F_2$.

Since there are no edges between $V(XQ_n^k) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a nature faulty set, $XQ_n^k - F_1$ has two parts $XQ_n^k - F_1 - F_2$ and $XQ_n^k[F_2 \setminus F_1]$ (for convenience). Thus, $\delta(XQ_n^k - F_1 - F_2) \geq 1$ and $\delta(XQ_n^k[F_2 \setminus F_1]) \geq 1$. Similarly, $\delta(XQ_n^k[F_1 \setminus F_2]) \geq 1$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a nature faulty set. When $F_1 \setminus F_2 = \emptyset$, $F_1 \cap F_2 = F_1$ is also a nature faulty set. Since there are no edges between $V(XQ_n^k - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a nature cut. By Theorem 9.2.4, $|F_1 \cap F_2| \geq 8n-4$. Note that $|F_2 \backslash F_1| \geq 2$. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq 2 + 8n - 4 = 8n-2$, which contradicts with that $|F_2| \leq 8n-3$.

So XQ_n^k is nature $(8n-3)$-diagnosable. By the definition of $t_1(XQ_n^k)$, $t_1(XQ_n^k) \geq 8n-3$.
□

Combining Lemmas 9.3.1 and 9.3.2, we have the following theorem.

Theorem 9.3.3 Let $n \geq 2$ and let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$. Then the nature diagnosability of XQ_n^k under the PMC model is $8n-3$.

9.4 The Nature Diagnosability of Expanded k-Ary n-Cubes under the MM* Model

In this section, we shall show the nature diagnosability of the expanded k-ary n-cube under the MM* model.

Lemma 9.4.1 Let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$. Then the nature diagnosability of XQ_n^k under the MM* model is less than or equal to $8n-3$, i.e., $t_1(XQ_n^k) \leq 8n-3$.

Proof: Let A, F_1 and F_2 be defined in Lemma 9.2.3(See Fig. 5.3). By the Lemma 9.2.3, $|F_1| = 8n-4$, $|F_2| = 8n-2$, $\delta(XQ_n^k - F_1) \geq 1$ and $\delta(XQ_n^k - F_2) \geq 1$. So both F_1 and F_2 are nature faulty sets. By the definitions of F_1 and F_2, $F_1 \triangle F_2 = A$. Note $F_1 \setminus F_2 = \emptyset$, $F_2 \setminus F_1 = A$ and $(V(XQ_n^k) \setminus (F_1 \cup F_2)) \cap A = \emptyset$. Therefore, both F_1 and F_2 are not satisfied with any condition in Theorem 5.3.2, and XQ_n^k is not nature $(8n-2)$-diagnosable. Hence, $t_1(XQ_n^k) \leq 8n-3$. □

Lemma 9.4.2 Let $n \geq 2$ and let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$. Then the nature diagnosability of XQ_n^k under the MM* model is more than or equal to $8n-3$, i.e., $t_1(XQ_n^k) \geq 8n-3$.

Proof: By the definition of nature diagnosability, it is sufficient to show that XQ_n^k is nature $(8n-3)$-diagnosable. By Theorem 5.3.2, suppose, on the contrary, that there are two distinct nature faulty subsets F_1 and F_2 of XQ_n^k with $|F_1| \leq 8n-3$ and $|F_2| \leq 8n-3$, but the vertex set pair (F_1, F_2) does not satisfy any condition in Theorem 5.3.2. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similarly to the discussion on $V(XQ_n^k) \neq F_1 \cup F_2$ in Lemma 9.3.2, we can deduce $V(XQ_n^k) \neq F_1 \cup F_2$. Therefore, $V(XQ_n^k) \neq F_1 \cup F_2$.

Claim 1. $XQ_n^k - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $XQ_n^k - F_1 - F_2$ has at least one isolated vertex w. Since F_1 is a nature faulty set, there is a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Since the vertex set pair (F_1, F_2) does not satisfy any condition in Theorem 5.3.2, there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Thus, there is just a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Assume $F_1 \setminus F_2 = \emptyset$. Then $F_1 \subseteq F_2$. Since F_2 is a nature faulty set, $XQ_n^k - F_2 = XQ_n^k - F_1 - F_2$ has no isolated vertex, a contradiction. Therefore, let $F_1 \setminus F_2 \neq \emptyset$ as follows. Similarly, we can deduce that there is just a vertex $v \in F_1 \setminus F_2$ such that v is adjacent to w. Let $W \subseteq V(XQ_n^k) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $XQ_n^k[V(XQ_n^k) \setminus (F_1 \cup F_2)]$, and let H be the subgraph induced by the vertex set $V(XQ_n^k) \setminus (F_1 \cup F_2 \cup W)$. Then for any $w \in W$, there are $(4n-2)$ neighbors in $F_1 \cap F_2$. Since $|F_2| \leq 8n-3$, we have $\sum_{w \in W} |N_{XQ_n^k[(F_1 \cap F_2) \cup W]}(w)| = |W|(4n-2) \leq \sum_{v \in F_1 \cap F_2} d_{XQ_n^k}(v) \leq |F_1 \cap F_2|(4n-2) \leq (|F_2|-1)(4n-2) \leq (8n-4)(4n-2) = 32n^2 - 32n + 8$. It follows that $|W| \leq \frac{32n^2-32n+8}{4n-2} \leq 8n-4$. Note $|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq 2(8n-3) - (4n-2) = 12n-4$. Suppose $V(H) = \emptyset$. Then $k^n = |V(XQ_n^k)| = |F_1 \cup F_2| + |W| \leq 12n-4+8n-4 = 20n-8$. This is a contradiction to $n \geq 2$. So $V(H) \neq \emptyset$. Since the vertex set pair (F_1, F_2) does not satisfy the condition (1) of Theorem 5.3.2, and any vertex of $V(H)$ is not isolated in H, we induce that there is no edge between $V(H)$ and $F_1 \triangle F_2$. Thus, $F_1 \cap F_2$ is a vertex cut of XQ_n^k and $\delta(XQ_n^k - (F_1 \cap F_2)) \geq 1$, i.e., $F_1 \cap F_2$ is a nature cut of XQ_n^k. By Theorem 9.2.4, $|F_1 \cap F_2| \geq 8n-4$. Because $|F_1| \leq 8n-3$, $|F_2| \leq 8n-3$, and neither $F_1 \setminus F_2$ nor $F_2 \setminus F_1$ is empty, we have $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$. Let $F_1 \setminus F_2 = \{v_1\}$ and $F_2 \setminus F_1 = \{v_2\}$. Then for any vertex $w \in W$, w are adjacent to v_1 and v_2. According to Proposition 9.1.6, there are at most three common neighbors for any pair of vertices in XQ_n^k when $k \geq 8$, it follows that there are at most two isolated vertices in $XQ_n^k - F_1 - F_2$, i.e., $|W| \leq 2$.

Suppose that there is exactly one isolated vertex v in $XQ_n^k - F_1 - F_2$. Let v_1 and v_2 be adjacent to v. Then $N_{XQ_n^k}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_1) \setminus \{v, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_2) \setminus \{v, v_1\} \subseteq F_1 \cap F_2$, $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, v_1\})| \leq 1$ and $|[N_{XQ_n^k}(v_1) \setminus \{v\}] \cap [N_{XQ_n^k}(v_2) \setminus \{v\}]| \leq 1$. Thus, $|F_1 \cap F_2| \geq |N_{XQ_n^k}(v) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(v_1) \setminus \{v, v_2\}| + |N_{XQ_n^k}(v_2) \setminus \{v, v_1\}| = (4n-2) + (4n-2) + (4n-2) - 3 = 12n-9$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 12n - 9 = 12n - 8 > 8n-3$ $(n \geq 2)$, which contradicts $|F_2| \leq 8n-3$.

Suppose that there are exactly two isolated vertices v and w in $XQ_n^k - F_1 - F_2$. Let v_1 and v_2 be adjacent to v and w, respectively. Then $N_{XQ_n^k}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(w) \setminus$

$\{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\} \subseteq F_1 \cap F_2$, $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| \leq 1$. $|(N_{XQ_n^k}(w) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(w) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| \leq 1$. By Proposition 9.1.6, there are at most two common neighbors for any pair of vertices in XQ_n^k. Thus, it follows that $|(N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| = 0$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(w) \setminus \{v_1, v_2\})| = 0$. Thus, $|F_1 \cap F_2| \geq |N_{XQ_n^k}(v) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(w) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\}| + |N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\}| = (4n-2) + (4n-2) + (4n-3) + (4n-3) - 1 - 1 - 1 - 1 = 16n - 14$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 16n - 14 = 16n - 13 > 8n - 3 \ (n \geq 2)$, which contradicts $|F_2| \leq 8n - 3$.

Suppose that $k = 6$, and v_1 and v_2 are adjacent. Proposition 9.1.6, $|N(v_1) \cap N(v_2)| \leq 2$. Therefore, $|W| \leq 2$.

Suppose that there is exactly one isolated vertex v in $XQ_n^k - F_1 - F_2$. Let v_1 and v_2 be adjacent to v. Then $N_{XQ_n^k}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_1) \setminus \{v, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_2) \setminus \{v, v_1\} \subseteq F_1 \cap F_2$, $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, v_1\})| \leq 1$ and $|[N_{XQ_n^k}(v_1) \setminus \{v\}] \cap [N_{XQ_n^k}(v_2) \setminus \{v\}]| \leq 1$. Thus, $|F_1 \cap F_2| \geq |N_{XQ_n^k}(v) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(v_1) \setminus \{v, v_2\}| + |N_{XQ_n^k}(v_2) \setminus \{v, v_1\}| = (4n-2) + (4n-2) + (4n-2) - 3 = 12n - 9$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 12n - 9 = 12n - 8 > 8n - 3 \ (n \geq 2)$, which contradicts $|F_2| \leq 8n - 3$.

Suppose that there are exactly two isolated vertices v and w in $XQ_n^k - F_1 - F_2$. Let v_1 and v_2 be adjacent to v and w, respectively. Then $N_{XQ_n^k}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(w) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\} \subseteq F_1 \cap F_2$, $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| \leq 1$. $|(N_{XQ_n^k}(w) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(w) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| \leq 1$. By Proposition 9.1.6, there are at most two common neighbors for any pair of vertices in XQ_n^k $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(w) \setminus \{v_1, v_2\})| = 0$. Thus, it follows that $|(N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| = 0$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(w) \setminus \{v_1, v_2\})| = 0$. Thus, $|F_1 \cap F_2| \geq |N_{XQ_n^k}(v) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(w) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\}| + |N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\}| = (4n-2) + (4n-2) + (4n-3) + (4n-3) - 1 - 1 -$

$1-1=16n-14$. It follows that $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\geq 1+16n-14=16n-13>8n-3$ $(n\geq 2)$, which contradicts $|F_2|\leq 8n-3$.

Suppose that $k=6$, and v_1 and v_2 are not adjacent. Proposition 9.1.6, $|N(v_1)\cap N(v_2)|\leq 4$ and hence $|W|\leq 4$. If $|N(v_1)\cap N(v_2)|=4$, then $v_1,v_2\in V(XQ[i])$. From Fig. 2.9 and 2.10, $XQ_1^6[N(v_1)\cap N(v_2)]$ is connected. Therefore, $|W|\leq 3$. Since $|N(v_1)\cap N(v_2)|\neq 3$, $|W|\leq 2$ holds.

Suppose that there is exactly one isolated vertex v in $XQ_n^k-F_1-F_2$. Let v_1 and v_2 be adjacent to v. Then $N_{XQ_n^k}(v)\setminus\{v_1,v_2\}\subseteq F_1\cap F_2$, $N_{XQ_n^k}(v_1)\setminus\{v\}\subseteq F_1\cap F_2$, $N_{XQ_n^k}(v_2)\setminus\{v\}\subseteq F_1\cap F_2$, $|(N_{XQ_n^k}(v)\setminus\{v_1,v_2\})\cap(N_{XQ_n^k}(v_1)\setminus\{v\})|\leq 2$ and $|(N_{XQ_n^k}(v)\setminus\{v_1,v_2\})\cap(N_{XQ_n^k}(v_2)\setminus\{v\})|\leq 2$ and $|[N_{XQ_n^k}(v_1)\setminus\{v\}]\cap[N_{XQ_n^k}(v_2)\setminus\{v\}]|\leq 3$. Thus, $|F_1\cap F_2|\geq |N_{XQ_n^k}(v)\setminus\{v_1,v_2\}|+|N_{XQ_n^k}(v_1)\setminus\{v\}|+|N_{XQ_n^k}(v_2)\setminus\{v\}|=(4n-2)+(4n-1)+(4n-1)-2-2-3=12n-11$. It follows that $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\geq 1+12n-11=12n-10>8n-3$ $(n\geq 2)$, which contradicts $|F_2|\leq 8n-3$.

Suppose that there are exactly two isolated vertices v and w in $XQ_n^k-F_1-F_2$. Let v_1 and v_2 be adjacent to v and w, respectively. Then $N_{XQ_n^k}(v)\setminus\{v_1,v_2\}\subseteq F_1\cap F_2$, $N_{XQ_n^k}(w)\setminus\{v_1,v_2\}\subseteq F_1\cap F_2$, $N_{XQ_n^k}(v_1)\setminus\{v,w\}\subseteq F_1\cap F_2$, $N_{XQ_n^k}(v_2)\setminus\{v,w\}\subseteq F_1\cap F_2$, $|(N_{XQ_n^k}(v)\setminus\{v_1,v_2\})\cap(N_{XQ_n^k}(v_1)\setminus\{v,w\})|\leq 2$ and $|(N_{XQ_n^k}(v)\setminus\{v_1,v_2\})\cap(N_{XQ_n^k}(v_2)\setminus\{v,w\})|\leq 2$. $|(N_{XQ_n^k}(w)\setminus\{v_1,v_2\})\cap(N_{XQ_n^k}(v_1)\setminus\{v,w\})|\leq 2$ and $|(N_{XQ_n^k}(w)\setminus\{v_1,v_2\})\cap(N_{XQ_n^k}(v_2)\setminus\{v,w\})|\leq 2$. By Proposition 9.1.6, there are at most four common neighbors for any pair of vertices in XQ_n^k. Thus, it follows that $|(N_{XQ_n^k}(v_1)\setminus\{v,w\})\cap(N_{XQ_n^k}(v_2)\setminus\{v,w\})|\leq 2$.

Thus, $|F_1\cap F_2|\geq |N_{XQ_n^k}(v)\setminus\{v_1,v_2\}|+|N_{XQ_n^k}(w)\setminus\{v_1,v_2\}|+|N_{XQ_n^k}(v_1)\setminus\{v,w\}|+|N_{XQ_n^k}(v_2)\setminus\{v,w\}|=(4n-2)+(4n-2)+(4n-2)+(4n-2)-2-2-2-2-2=16n-18$. It follows that $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\geq 1+16n-18=16n-17>8n-3$ $(n\geq 2)$, which contradicts $|F_2|\leq 8n-3$.

The proof of Claim 1 is completed.

Let $u\in V(XQ_n^k)\setminus(F_1\cup F_2)$. By Claim 1, u has at least one neighbor in $XQ_n^k-F_1-F_2$. Since the vertex set pair (F_1,F_2) does not satisfy any condition in Theorem 5.3.2, by the condition (1) of Theorem 5.3.2, for any pair of adjacent vertices $u,w\in V(XQ_n^k)\setminus(F_1\cup F_2)$, there is no vertex $v\in F_1\triangle F_2$ such that $uw\in E(XQ_n^k)$ and $vw\in E(XQ_n^k)$. It follows that u has no neighbor in $F_1 \triangle F_2$. By the arbitrariness of u, there is no edge between $V(XQ_n^k) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a nature faulty set, $\delta_{XQ_n^k}([F_2 \setminus F_1]) \geq 1$ and hence $|F_2 \setminus F_1| \geq 2$. Since both F_1 and F_2 are nature faulty sets, and there is no edge between $V(XQ_n^k) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a nature cut of XQ_n^k. By Theorem 9.2.4, we have $|F_1 \cap F_2| \geq 8n-4$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 2 + (8n-4) = 8n-2$, which contradicts $|F_2| \leq 8n-3$. Therefore, XQ_n^k is nature $(8n-3)$-diagnosable and $t_1(XQ_n^k) \geq 8n-3$. The proof is completed. □

Combining Lemmas 9.4.1 and 9.4.2, we have the following theorem.

Theorem 9.4.3 Let $n \geq 2$. Then the nature diagnosability of the expanded k-ary n-cube XQ_n^k under the MM* model is $8n-3$.

9.5 Conclusion

In this chapter, we investigated the problem of the nature diagnosability of the expanded k-ary n-cube XQ_n^k under the PMC model and MM*. As we discussed in Chapter 2, expanded k-ary n-cube XQ_n^k is a generalization of k-ary n-cube. The results in this chapter provide a solid base for further investigation on connectivity and diagnosability of expanded k-ary n-cube XQ_n^k.

Chapter 10

The Tightly Super-3-extra Connectivity & Diagnosability of Locally Twisted Cubes

In this chapter, we show that LTQ_n is tightly $(4n-9)$ super-3-extra-connected for $n \geq 6$ and the 3-extra diagnosability of LTQ_n under the PMC model and MM* model is $4n-6$ for $n \geq 5$ and $n \geq 7$, respectively. The results in this chapter is published in American Journal of Computational Mathematics [88].

10.1 The Connectivity of Locally Twisted Cubes

Firstly we will list some known results on the structure of LTQ_n which are useful for the investigation.

Proposition 10.1.1 [72] Let LTQ_n be the locally twisted cube. If two vertices u, v are adjacent, then there is no common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| = 0$. If two vertices u, v are not adjacent, then there are at most two common neighbor vertices of these two vertices, i.e., $|N(u) \cap N(v)| \leq 2$.

Lemma 10.1.1 [109] Let LTQ_n be the locally twisted cube. Then $\kappa(LTQ_n) = n$.

Lemma 10.1.2 [33] Let LTQ_n be the locally twisted cube, and let $S \subseteq V(LTQ_n)$ and $n \geq 3$. If $LTQ_n - S$ is disconnected and $n \leq |S| \leq 2n-3$, then $LTQ_n - S$ has exactly two components, one is trivial and the other is nontrivial.

Lemma 10.1.3 [73] Let LTQ_n be the locally twisted cube. Then all cross-edges of LTQ_n is a perfect matching.

Lemma 10.1.4 [43] Let LTQ_n be the locally twisted cube. Then $\kappa^{(2)}(LTQ_n) = 4n-8$.

For any four vertices in LTQ_n, it is easy to have that there are only three different $LTQ_n[\{u,v,w,x\}]$'s: a 3-path, a graph isomorphic to $K_{1,3}$ and 4-cycle. Based on this, we could investigate the $N(V(LTQ_n[\{u,v,w,x\}]))$ and its cardinality.

Lemma 10.1.5 Let LTQ_n be the locally twisted cube. If $P = uvwx$ is a 3-path in LTQ_n and $ux \notin E(LTQ_n)$ for $n \geq 3$, then $|N(V(P))| \geq 4n-9$.

Proof: We decompose LTQ_n into $0LTQ_{n-1}$ and $1LTQ_{n-1}$. Then $0LTQ_{n-1}$ and $1LTQ_{n-1}$ are isomorphic to LTQ_{n-1}. Without loss of generality, we have the following cases.

Case 1 . $u,x \in V(0LTQ_{n-1})$ and $v,w \in V(1LTQ_{n-1})$.

Since $u \in V(0LTQ_{n-1})$, $v \in V(1LTQ_{n-1})$ and u,v are adjacent, by Proposition 10.1.1, u,v have no common neighbor vertices. Similarly, x,w have no common neighbor vertices and v,w have no common neighbor vertices. Since $u \in V(0LTQ_{n-1})$, $w \in V(1LTQ_{n-1})$, u,w are not adjacent, v is a common neighbor vertex of u,w, $x \in V(0LTQ_{n-1})$ and x is a neighbor vertex of w, by Lemma 10.1.3, $|(N(u) \cap N(w)) \setminus \{v\}| = 0$. Similarly, $|(N(x) \cap N(v)) \setminus \{w\}| = 0$. Since u and x are not adjacent, by proposition 10.1.1, $|N(u) \cap N(x)| \leq 2$. Therefore, $|N(V(P))| \geq 2(n-1) + 2(n-2) - 2 = 4n-8$.

Case 2 . $u \in V(0LTQ_{n-1})$ and $v,w,x \in V(1LTQ_{n-1})$.

Since u,v are adjacent, by Proposition 10.1.1, $|N(u) \cap N(v)| = 0$. Similarly, $|N(v) \cap N(w)| = 0$, $|N(x) \cap N(w)| = 0$. And since $u \in V(0LTQ_{n-1})$, $w \in V(1LTQ_{n-1})$, u,w are not adjacent and v is the common neighbor vertex of u and w, by Lemma 10.1.3, $|(N(u) \cap N(w)) \setminus \{v\}| \leq 1$. Since u,x are not adjacent, $u \in V(0LTQ_{n-1})$, $x \in V(1LTQ_{n-1})$, by Lemma 10.1.3, $|N(u) \cap N(x)| \leq 1$. Since w is the common neighbor vertex of v and x and v,x

are not adjacent, by proposition 10.1.1, $|(N(v)\cap N(x))\backslash\{w\}|\leq 1$. Therefore, $|N(P)|\geq 2(n-1)+2(n-2)-3=4n-9$.

Case 3 . $u,v\in V(0LTQ_{n-1})$ and $w,x\in V(1LTQ_{n-1})$.

Since u,v are adjacent, by Proposition 10.1.1, $|N(u)\cap N(v)|=0$. Similarly, $|N(v)\cap N(w)|=0$, $|N(w)\cap N(x)|=0$. Since $u\in V(0LTQ_{n-1})$, $x\in V(1LTQ_{n-1})$ and u, x are not adjacent, by proposition 10.1.1, $|N(u)\cap N(x)|\leq 2$. If $|(N(u)\cap N(w))\setminus\{v\}|=1$, then, by Lemma 10.1.3, $|N(u)\cap N(x)|\leq 1$. If $|(N(u)\cap N(w))\setminus\{v\}|=0$, then, by Lemma 10.1.3, $|N(u)\cap N(x)|\leq 2$. Therefore, $|N(V(P))|\geq 2(n-1)+2(n-2)-2=4n-8$.

In conclusion, $|N(V(P))|\geq 4n-9$. □

As follow we have another structure, which is formed by four vertices. We investigate the $N(V(LTQ_n[\{u,v,w,x\}]))$ and its cardinality.

Lemma 10.1.6 Let LTQ_n be the locally twisted cube. If $LTQ_n[\{u,v,w,x\}]$ is isomorphic to $K_{1,3}$ for $n\geq 3$ and $d(u)=3$, then $|N(V(LTQ_n[\{u,v,w,x\}]))|\geq 4n-9$.

Proof: Since $d(u)=3$ and $LTQ_n[\{u,v,w,x\}]$ is isomorphic to $K_{1,3}$, we have $d(v)=1$, $d(w)=1$ and $d(x)=1$. Since v,w are not adjacent and u is a common neighbor vertex of v, w, by Proposition 10.1.1, $(|N(v)\cap N(w))\backslash\{u\}|\leq 1$. Similarly, $|(N(v)\cap N(x))\setminus\{u\}|\leq 1$, $|(N(w)\cap N(x))\setminus\{u\}|\leq 1$. Therefore, $|N(V(LTQ_n[\{u,v,w,x\}]))|\geq 3(n-1)+(n-3)-3=4n-9$. □

If $LTQ_n[\{u,v,w,x\}]$ is a 4-cycle, then $|N(V(LTQ_n[\{u,v,w,x\}]))|=4n-8$. Combining this with Lemmas 10.1.5 and 10.1.6, we have the following corollary.

Corollary 10.1.7 Let LTQ_n be the locally twisted cube and let H be a connected subgraph of LTQ_n. If $|V(H)|\geq 4$, then $|N(V(H))|\geq 4n-9$.

Here we prove a lemma as follow since it will be used in the proofs to find the lower bounds of the 3-extra diagnosability of LTQ_n under the PMC model and MM* model, where $n\geq 4$.

Lemma 10.1.8 Let $A=\{0\cdots 0001, 0\cdots 0111, 0\cdots 0101, 0\cdots 0100\}$ and let LTQ_n be the locally twisted cube with $n\geq 4$. If $F_1=N_{LTQ_n}(A)$, $F_2=F_1\cup A$, where $n\geq 4$, then $|F_1|=$

$4n-9$, $|F_2| = 4n-5$, F_1 is a 3-extra cut of LTQ_n, $LTQ_n - F_1$ has two components $LTQ_n - F_2$ and $LTQ_n[A]$, $|V(LTQ_n - F_2)| \geq 4$, and $|A| \geq 4$.

Proof: According to the definition, $LTQ_n[A]$ is a 3-path and $|A| = 4$. By Lemma 10.1.5, $|F_1| \geq 4n-9$. From Fig. 2.11, we have $|F_1| = 3$. By the definition of LTQ_n, $|F_1| = 3+4(n-3) = 4n-9$. Therefore, $|F_2| = |F_1| + |A| = (4n-9)+4 = 4n-5$. Let $F_2^i = V(iLTQ_{n-1}) \cap F_2$, $i \in \{0,1\}$.

To prove $LTQ_n - F_2$ has two components and $|V(LTQ_n - F_2)| \geq 4$, we first claim the following.

Claim 1 . $LTQ_n - F_2$ is connected for $n \geq 4$.

we prove by induction on n. For $n = 4$, $A = \{0001, 0111, 0101, 0100\}$, $F_1 = \{0000, 0011, 0110, 1001, 1011, 1101, 1100\}$. It is easy to see that $LTQ_4 - F_2$ is connected (See Fig. 2.12). When $n = 5$, $A = \{00001, 00111, 00101, 00100\}$, $F_2^1 = \{11001, 11110, 11111, 10100\}$ (See Fig. 2.13). It is clear that $1LTQ_{n-1} - F_2^1$ is connected (See Fig. 2.13). We discompose LTQ_n into $0LTQ_{n-1}$ and $1LTQ_{n-1}$. Assume that $n \geq 6$, the result holds for LTQ_{n-1}. Then $0LTQ_{n-1} - F_2^0$ is connected. Note that $A \subseteq V(0LTQ_{n-1})$ and $|N(A) \cap V(1LTQ_{n-1})| = 4$.nBy Lemma 10.1.1, $1LTQ_{n-1} - F_2^1$ is connected. By inductive hypothesis, $0LTQ_{n-1} - F_2^0$ is connected. Since $2^{n-1} > 4n-5$, by Lemma 10.1.3, $LTQ_n - F_2$ is connected. The proof of Claim 1 is completed.

By Claim 1, $LTQ_n - F_1$ has two components $LTQ_n - F_2$ and $LTQ_n[A]$ for $n \geq 4$. Then $|V(LTQ_n - F_2)| = 2^n - (4n-5) \geq 4$ for $n \geq 4$. And since $|A| = 4$, F_1 is a 3-extra cut of LTQ_n. $\square$

In order to prove that LTQ_n is tightly (4n-9) super-3-extra-connected, which will be indispensable part in the proof to show the 3-extra diagnosability of LTQ_n under the MM* model, we prove the following 2 lemmas and show an existing theorem and an existing lemma, where $n \geq 6$.

The number of different cases of $LTQ_n - F$ varies according to the different choice of the interval of $|F|$, based on this, we divide $|F|$ into two intervals: $|F| \leq 3n-6$ and $3n-5 \leq |F| \leq 4n-10$, where $n \geq 5$. We firstly list the result in the first interval $|F| \leq 3n-6$.

Lemma 10.1.9 [73] Let $LTQ_n (n \geq 4)$ be the locally twisted cube. If $|F| \leq 3n-6$, then $LTQ_n - F$ satisfies one of the following conditions:

(1) $LTQ_n - F$ has three components, two of which are isolated vertices;

(2) $LTQ_n - F$ has two components, one of which is an isolated vertex;

(3) $LTQ_n - F$ has two components, one of which is a K_2;

(4) $LTQ_n - F$ is connected.

Theorem 10.1.10 [118] Let LTQ_n be the locally twisted cube. Then $\tilde{\kappa}^{(3)}(LTQ_n) = 4n-9$ for $n \geq 4$.

Here we specially pick up the case that $|F| = 10$ for $n = 5$ as the following lemma to facilitate the understanding of Lemma 10.1.12.

Lemma 10.1.11 Let LTQ_n be the locally twisted cube. If $|F| = 10$ for $n = 5$, then $LTQ_5 - F$ satisfies one of the following conditions:

(1) $LTQ_5 - F$ has four components, three of which are isolated vertices;

(2) $LTQ_5 - F$ has three components, one of which is isolated vertices and one of which is a K_2;

(3) $LTQ_5 - F$ has three components, two of which are isolated vertices;

(4) $LTQ_5 - F$ has two components, one of which is a path of length two;

(5) $LTQ_5 - F$ has two components, one of which is an isolated vertex;

(6) $LTQ_5 - F$ has two components, one of which is a K_2;

(7) $LTQ_5 - F$ is connected.

Proof: We decompose LTQ_5 into $0LTQ_4$ and $1LTQ_4$. Then $0LTQ_4$ and $1LTQ_4$ are isomorphic to LTQ_4. Suppose that $F_i = F \cap V(iLTQ_4)$, $i \in \{0,1\}$. Without loss of generality, let $|F_0| \geq |F_1|$. And since $|F| = 10$, $5 \leq |F_0| \leq 10$, $0 \leq |F_1| \leq 5$. Let C_i be the maximum component of $iLTQ_4 - F_i$, $i \in \{0,1\}$. We consider the following cases.

Case 1 . $|F_0| = 5$.

Since $|F_0| = 5$ and $|F| = 10$, $|F_1| = 10 - 5 = 5$. By Lemmas 10.1.1 and 10.1.2, both $0LTQ_4 - F_0$ and $1LTQ_4 - F_1$ are connected or has two components, one of which is an isolated vertex. Since $2^{5-1}-6-2 \geq 1$, by Lemma 10.1.3, $LTQ_n[V(C_0) \cup V(C_1)]$ is connected. Thus, $LTQ_5 - F$ satisfies one of conditions:

(1) $LTQ_5 - F$ has three components, two of which are isolated vertices;

(2) $LTQ_5 - F$ has two components, one of which is an isolated vertex;

(3) $LTQ_5 - F$ has two components, one of which is a K_2;

(4) $LTQ_5 - F$ is connected.

Case 2 . $|F_0| = 6$.

Since $|F_0| = 6$ and $|F| = 10$, $|F_1| = 10-6 = 4$. By Lemmas 10.1.1 and 10.1.2, $1LTQ_4 - F_1$ is connected or has two components, one of which is an isolated vertex. Since $|F_0| = 6$, by Lemma 10.1.9, $0LTQ_4 - F_0$ satisfies one of the following conditions:

(1) $0LTQ_4 - F_0$ has three components, two of which are isolated vertices;

(2) $0LTQ_4 - F_0$ has two components, one of which is an isolated vertex;

(3) $0LTQ_4 - F_0$ has two components, one of which is a K_2;

(4) $0LTQ_4 - F_0$ is connected.

Then $LTQ_5 - F$ satisfies one of the conditions (1)-(7).

Case 3 . $|F_0| \geq 7$.

Since $|F_0| \geq 7$ and $|F| = 10$, $|F_1| \leq 10-7 = 3$. By Lemma 10.1.1, $1LTQ_4 - F_1$ is connected.

Suppose that $0LTQ_4 - F_0$ is connected. Since $2^{5-1} - 10 \geq 1$, by Lemma 10.1.3, $LTQ_n - F$ is connected.

Suppose that $0LTQ_4 - F_0$ is not connected. Let the components in $0LTQ_4 - F_0$ be G_1, $G_2, \ldots, G_k$ for $k \geq 2$ and $|V(G_1)| \leq |V(G_2)| \leq \ldots \leq |V(G_k)|$. If $|V(G_r)| \geq 4 (1 \leq r \leq k-1)$, by Lemma 10.1.3, $|N(V(G_r)) \cap V(1LTQ_4)| \geq 4$. Combining this with $|F_1| \leq 3$, we have that $LTQ_5[V(G_r) \cup V(1LTQ_4 - F_1)]$ is connected. Therefore, G_r is not a component of $LTQ_5 - F$ for $|V(G_r)| \geq 4$. Therefore, $LTQ_5 - F$ is connected. The following we discuss G_r is a component of $LTQ_5 - F$ with $|V(G_r)| \leq 3 (1 \leq r \leq k-1)$.

If $k = 5$, by Lemma 10.1.3, $|N(V(G_1)) \cup N(V(G_2)) \cup \ldots \cup N(V(G_{k-1})) \cap V(1LTQ_4)| \geq 4$. Combining this with $|F_1| \leq 3$, there is one $G_r (1 \leq r \leq k-1)$ such that $LTQ_5[V(G_r) \cup$

$V(1LTQ_4 - F_1)]$ is connected. Thus, $k \le 4$. Since $|F_1| = 3$, $k \le 4$, and $|V(G_r)| \le 3 (1 \le r \le k-1)$, $LTQ_5 - F$ satisfies one of the conditions (1)-(7). $\square$

Lemma 10.1.12 Let LTQ_n be the locally twisted cube. If $3n-5 \le |F| \le 4n-10$ for $n \ge 5$, then $LTQ_n - F$ satisfies one of the following conditions:

(1) $LTQ_n - F$ has four components, three of which are isolated vertices;

(2) $LTQ_n - F$ has three components, one of which is isolated vertices and one of which is a K_2;

(3) $LTQ_n - F$ has three components, two of which are isolated vertices;

(4) $LTQ_n - F$ has two components, one of which is a path of length two;

(5) $LTQ_n - F$ has two components, one of which is an isolated vertex;

(6) $LTQ_n - F$ has two components, one of which is a K_2;

(7) $LTQ_n - F$ is connected.

Proof: By Lemma 10.1.11, the result holds for $n = 5$. We proceed by induction on n. Assume $n \ge 6$ and the result holds for LTQ_{n-1}, i.e., if $3n-5 \le |F| \le 4(n-1)-10 = 4n-14$, then $LTQ_{n-1} - F$ satisfies one of the conditions (1)-(7) in Lemma 10.1.12. The following we prove $LTQ_n - F$ satisfies one of the conditions (1)-(7).

We decompose LTQ_n into $0LTQ_{n-1}$ and $1LTQ_{n-1}$. Then $0LTQ_{n-1}$ and $1LTQ_{n-1}$ are isomorphic to LTQ_{n-1}. Suppose that $F_i = F \cap V(iLTQ_{n-1})$, $i \in \{0,1\}$. Without loss of generality, let $|F_0| \ge |F_1|$. And since $3n-5 \le |F| \le 4n-10$, $n \le \lceil \frac{3n-5}{2} \rceil \le |F_0| \le 4n-10$, $0 \le |F_1| \le \lfloor \frac{4n-10}{2} \rfloor \le 2n-5$. Let C_i be the maximum component of $iLTQ_{n-1} - F_i$, $i \in \{0,1\}$. We consider the following cases.

Case 1 . $n \le |F_0| \le 3(n-1)-6 = 3n-9$.

Since $|F_0| \ge |F_1|$ and $|F| \le 4n-10$, $(4n-10)-(3n-9) = n-1 \le |F_1| \le \lfloor \frac{4n-10}{2} \rfloor = 2n-5$. By Lemmas 10.1.1 and 10.1.2, $1LTQ_{n-1} - F_1$ is connected or has two components, one of which is an isolated vertex. Since $n \le |F_0| \le 3(n-1)-6 = 3n-9$, by lemma 10.1.9, $0LTQ_{n-1} - F_0$ satisfies one of the following conditions:(1) $0LTQ_{n-1} - F_0$ has three components, two of which are isolated vertices; (2) $0LTQ_{n-1} - F_0$ has two components, one of which is an isolated vertex; (3) $0LTQ_{n-1} - F_0$ has two components, one of which is a K_2; (4) $0LTQ_{n-1}-F_0$ is connected. Since $2^{n-1}-(4n-10)-3 \geq 1$, by Lemma 10.1.3, $LTQ_n[V(C_0)\cup V(C_1)]$ is connected. Thus, LTQ_n-F satisfies one of conditions (1)-(7) in Lemma 10.1.12.

Case 2 . $3n-8 \leq |F_0| \leq 4n-14$.

Since $|F_0| \geq |F_1|$ and $|F| \leq 4n-10$, $|F_1| \leq (4n-10)-(3n-8) = n-2$. By Lemma 10.1.1 , $1LTQ_{n-1}-F_1$ is connected. Since $3n-8 \leq |F_0| \leq 4n-14$, according to inductive hypothesis, $0LTQ_{n-1}-F_0$ satisfies one of the following conditions:

(1) $0LTQ_{n-1}-F_0$ has four components, three of which are isolated vertices;

(2) $0LTQ_{n-1}-F_0$ has three components, one of which is isolated vertices and one of which is a K_2;

(3) $0LTQ_{n-1}-F_0$ has three components, two of which are isolated vertices;

(4) $0LTQ_{n-1}-F_0$ has two components, one of which is a path of length two;

(5) $0LTQ_{n-1}-F_0$ has two components, one of which is an isolated vertex;

(6) $0LTQ_{n-1}-F_0$ has two components, one of which is a K_2;

(7) $0LTQ_{n-1}-F_0$ is connected.

Thus, LTQ_n-F satisfies one of the conditions (1)-(7) in Lemma 10.1.12.

Case 3 . $4n-13 \leq |F_0| \leq 4n-10$.

Since $4n-13 \leq |F_0| \leq 4n-10$ and $|F| \leq 4n-10$, $|F_1| \leq (4n-10)-(4n-13) = 3$. By Lemma 10.1.1, $1LTQ_{n-1}-F_1$ is connected.

Suppose that $0LTQ_{n-1}-F_0$ is connected. Since $2^{n-1}-(4n-10) \geq 1$, by Lemma 10.1.3, LTQ_n-F is connected.

Suppose that $0LTQ_{n-1}-F_0$ is not connected. Let the components in $0LTQ_{n-1}-F_0$ be G_1, $G_2,\ldots,G_k$ for $k \geq 2$ and $|V(G_1)| \leq |V(G_2)| \leq \ldots \leq |V(G_k)|$. If $|V(G_r)| \geq 4(1 \leq r \leq k-1)$, by Lemma 10.1.3, $|N(V(G_r))\cap V(1LTQ_{n-1})| \geq 4$. Combining this with $|F_1| \leq (4n-10)-(4n-13) = 3$, we have that $LTQ_n[V(G_r)\cup V(1LTQ_{n-1}-F_1)]$ is connected. Therefore, G_r is not a component of LTQ_n-F for $|V(G_r)| \geq 4$. Therefore, LTQ_n-F is connected. The following we discuss G_r is a component of LTQ_n-F with $|V(G_r)| \leq 3(1 \leq r \leq k-1)$.

If $k=5$, by Lemma 10.1.3, $|N(V(G_1))\cup N(V(G_2))\cup\ldots\cup N(V(G_{k-1}))\cap V(1LTQ_{n-1})| \geq 4$. Combining this with $|F_1| \leq 3$, there is one $G_r(1 \leq r \leq k-1)$ such that $LTQ_n[V(G_r)\cup$

$V(1LTQ_{n-1} - F_1)]$ is connected. Thus, $k \leq 4$. Since $|F_1| \leq 3$, $|V(G_r)| \leq 3(1 \leq r \leq k-1)$ and $k \leq 4$, $LTQ_n - F$ satisfies one of the conditions (1)-(7). □

Based on the above Lemma 10.1.9, Lemma 10.1.11, Lemma 10.1.12 and Theorem 10.1.10, we start to prove the following theorem.

Theorem 10.1.13 Let LTQ_n be the locally twisted cube for $n \geq 6$. Then LTQ_n is tightly $(4n-9)$ super-3-extra-connected.

Proof: By Theorem 10.1.10, we know for any minimum 3-extra cut $F \subset V(LTQ_n)$, $|F| = 4n-9$. We decompose LTQ_n into $0LTQ_{n-1}$ and $1LTQ_{n-1}$. Then $0LTQ_{n-1}$ and $1LTQ_{n-1}$ are isomorphic to LTQ_{n-1}. Suppose that $F_i = F \cap V(iLTQ_{n-1})$, $i \in \{0,1\}$. Without loss of generality, let $|F_0| \geq |F_1|$. And since $|F| = 4n-9$, $2n-4 \leq \lceil \frac{4n-9}{2} \rceil \leq |F_0| \leq 4n-9$, $0 \leq |F_1| \leq \lfloor \frac{4n-9}{2} \rfloor \leq 2n-5$. Let C_i be the maximum component of $iLTQ_{n-1} - F_i$, $i \in \{0,1\}$. We consider the following cases.

Case 1 . $2n-4 \leq |F_0| \leq 3(n-1)-6 = 3n-9$.

Since $|F_0| \geq |F_1|$ and $|F| = 4n-9$, $|F_1| \leq 2n-5$ holds.

By Lemmas 10.1.1 and 10.1.2, $1LTQ_{n-1} - F_1$ is connected or has two components, one of which is an isolated vertex. Since $2n-4 \leq |F_0| \leq 3(n-1)-6 = 3n-9$, by lemma 10.1.9, $0LTQ_{n-1} - F_0$ satisfies one of the following conditions: (1) $0LTQ_{n-1} - F_0$ has three components, two of which are isolated vertices; (2) $0LTQ_{n-1} - F_0$ has two components, one of which is an isolated vertex; (3) $0LTQ_{n-1} - F_0$ has two components, one of which is a K_2; (4) $0LTQ_{n-1} - F_0$ is connected. Since $2^{n-1} - (4n-9) - 3 \geq 1$, by Lemma 10.1.3, $LTQ_n[V(C_0) \cup V(C_1)]$ is connected. Then $LTQ_n - F$ satisfies one of the following conditions:

(1) $LTQ_n - F$ has four components, three of which are isolated vertices;

(2) $LTQ_n - F$ has three components, one of which is isolated vertices and one of which is a K_2;

(3) $LTQ_n - F$ has three components, two of which are isolated vertices;

(4) $LTQ_n - F$ has two components, one of which is a path of length two;

(5) $LTQ_n - F$ has two components, one of which is an isolated vertex;

(6) $LTQ_n - F$ has two components, one of which is a K_2;

(7) $LTQ_n - F$ is connected.

Thus, in this case, F is not a minimum 3-extra cut of LTQ_n, a contradiction.

Case 2 . $|F_0| = 3n-8$.

Since $|F_0| = 3n-8$ and $|F| = 4n-9$, we have $|F_1| = (4n-9)-(3n-8) = n-1$. By Lemmas 10.1.1 and 10.1.2, $1LTQ_{n-1} - F_1$ is connected or has two components, one of which is an isolated vertex. Since $|F_0| = 3n-8$, by Lemma 10.1.12, $0LTQ_{n-1} - F_0$ satisfies one of the following conditions:

(1) $0LTQ_{n-1} - F_0$ has four components, three of which are isolated vertices;

(2) $0LTQ_{n-1} - F_0$ has three components, one of which is isolated vertices and the other of which is a K_2;

(3) $0LTQ_{n-1} - F_0$ has three components, two of which are isolated vertices;

(4) $0LTQ_{n-1} - F_0$ has two components, one of which is a path of length two;

(5) $0LTQ_{n-1} - F_0$ has two components, one of which is an isolated vertex;

(6) $0LTQ_{n-1} - F_0$ has two components, one of which is a K_2;

(7) $0LTQ_{n-1} - F_0$ is connected.

If $0LTQ_{n-1} - F_0$ satisfies the condition (4), i.e., $0LTQ_{n-1} - F_0$ has two components, one of which is a path of length two, denoted by $P = uvw$, $1LTQ_{n-1} - F_1$ has two components, one of which is an isolated vertex x, and $|N(x) \cap V(P)| = 1$, $(N(V(P)) \cap V(1LTQ_{n-1})) \setminus \{x\} \subseteq F_1$, then, by Lemma 10.1.3, $LTQ_n - F$ has one component which is a 3-path or a $K_{1,3}$. Since $2^{n-1} - (4n-9) - 3 \geq 1$ for $n \geq 6$, $LTQ_n[C_0 \cup C_1]$ is connected. Thus, $LTQ_n - F$ exactly has two components. Then the other component C satisfies $|C| = 2^n - (4n-9) - 4 > 4$ for $n \geq 6$. Otherwise, F is not a minimum 3-extra cut of LTQ_n, a contradiction.

Case 3 . $3n-7 \leq |F_0| \leq 4n-14$.

Since $|F_0| \geq |F_1|$ and $|F| \leq 4n-9$, $|F_1| \leq (4n-9)-(3n-7) = n-2$. By Lemma 10.1.1, $1LTQ_{n-1} - F_1$ is connected. Since $3n-7 \leq |F_0| \leq 4n-14$, by Lemma 10.1.12, $0LTQ_{n-1} - F_0$ satisfies one of the following conditions:

(1) $0LTQ_{n-1} - F_0$ has four components, three of which are isolated vertices;

(2) $0LTQ_{n-1} - F_0$ has three components, one of which is isolated vertices and the other of which is a K_2;

(3) $0LTQ_{n-1} - F_0$ has three components, two of which are isolated vertices;

(4) $0LTQ_{n-1} - F_0$ has two components, one of which is a path of length two;

(5) $0LTQ_{n-1} - F_0$ has two components, one of which is an isolated vertex;

(6) $0LTQ_{n-1} - F_0$ has two components, one of which is a K_2;

(7) $0LTQ_{n-1} - F_0$ is connected.

Thus, $LTQ_n - F$ satisfies one of the following conditions:

(1) $LTQ_n - F$ has four components, three of which are isolated vertices;

(2) $LTQ_n - F$ has three components, one of which is isolated vertices and one of which is a K_2;

(3) $LTQ_n - F$ has three components, two of which are isolated vertices;

(4) $LTQ_n - F$ has two components, one of which is a path of length two;

(5) $LTQ_n - F$ has two components, one of which is an isolated vertex;

(6) $LTQ_n - F$ has two components, one of which is a K_2;

(7) $LTQ_n - F$ is connected.

In this case, F is not a minimum 3-extra cut of LTQ_n, a contradiction.

Case 4 . $|F_0| = 4n-13$.

Since $|F_0| = 4n-13$ and $|F| = 4n-9$ for $n \geq 6$, $|F_1| = (4n-9)-(4n-13) = 4$. By Lemma 10.1.1, $1LTQ_{n-1} - F_1$ is connected.

If there exists a 3-path P in $0LTQ_{n-1} - F_0$, then $N(V(P)) \cap V(0LTQ_{n-1}) \subseteq F_0$. By Corollary 10.1.7, $|N(V(P))| \geq 4n-13 = |F_0|$ in $0LTQ_{n-1} - F_0$. Therefore, $N(V(P)) = F_0$ in $0LTQ_{n-1} - F_0$. Note that $2^{n-1} - (4n-9) - 4 \geq 1$ for $n \geq 6$, by Lemma 10.1.3, then $LTQ_n[V(C_0) \cup V(C_1)]$ is connected. Then $LTQ_n - F$ just has two components, one of which is a 3-path.

If there exists a component $K_{1,3}$ in $0LTQ_{n-1} - F_0$, then $N_{0LTQ_{n-1}}(V(K_{1,3})) \subseteq F_0$. By Corollary 10.1.7, $|N(V(K_{1,3}))| \geq 4n-13 = |F_0|$ in $0LTQ_{n-1} - F_0$. Therefore, $N(V(K_{1,3})) = F_0$ in $0LTQ_{n-1} - F_0$. Note that $2^{n-1} - (4n-9) - 4 \geq 1$ for $n \geq 6$, by Lemma 10.1.3, $LTQ_n - F$ just has two components, one of which is a $K_{1,3}$.

If there exists a 4-cycle C in $0LTQ_{n-1}-F_0$, then $N_{0LTQ_{n-1}}(C)\cap V(0LTQ_{n-1})\subseteq F_0$. By Proposition 10.1.1, $|N_{0LTQ_{n-1}}(V(C))|\geq 4(n-1-2)=4n-12>4n-13=|F_0|$, a contradiction to $|F_0|=4n-13$. Therefore, $0LTQ_{n-1}-F_0$ has not a 4-cycle.

Case 5 . $4n-12\leq |F_0|\leq 4n-9$.

Since $4n-12\leq |F_0|\leq 4n-9$ and $|F|\leq 4n-9$, $|F_1|\leq (4n-9)-(4n-12)=3$. By Lemma 10.1.1, $1LTQ_{n-1}-F_1$ is connected.

Suppose that $0LTQ_{n-1}-F_0$ is connected. Since $2^{n-1}-(4n-9)\geq 1$, by Lemma 10.1.3, LTQ_n-F is connected, a contradiction.

Suppose that $0LTQ_{n-1}-F_0$ is not connected. Let the components in $0LTQ_{n-1}-F_0$ be G_1, G_2, …, G_k for $k\geq 2$ and $|V(G_1)|\leq |V(G_2)|\leq \ldots \leq |V(G_k)|$. If $|V(G_r)|\geq 4(1\leq r\leq k-1)$, by Lemma 10.1.3, $|N(V(G_r))\cap V(1LTQ_{n-1})|\geq 4$. If $k\geq 5$, by Lemma 10.1.3, $|N(V(G_1))\cup N(V(G_2))\cup\ldots\cup N(V(G_{k-1}))\cap V(1LTQ_{n-1})|\geq 4$. Combining this with $|F_1|\leq (4n-9)-(4n-12)=3$, we have that LTQ_n-F satisfies one of the following conditions:

(1) LTQ_n-F has four components, three of which are isolated vertices;

(2) LTQ_n-F has three components, one of which is isolated vertices and one of which is a K_2;

(3) LTQ_n-F has three components, two of which are isolated vertices;

(4) LTQ_n-F has two components, one of which is a path of length two;

(5) LTQ_n-F has two components, one of which is an isolated vertex;

(6) LTQ_n-F has two components, one of which is a K_2;

(7) LTQ_n-F is connected.

In this case, F is not a minimum 3-extra cut of LTQ_n, a contradiction. □

10.2 The 3-Extra Diagnosability of the Locally Twisted Cube under the PMC Model

In this section, we shall show the 3-extra diagnosability of locally twisted cubes under the PMC model.

Here we give the necessary and sufficient condition of that a system (graph) G is g-extra t-diagnosable under PMC model.

Theorem 10.2.1 [25, 112, 114] A system $G = (V,E)$ is g-extra t-diagnosable under the PMC model if and only if there is an edge $uv \in E$ with $u \in V \setminus (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of g-extra faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$.

Lemma 10.2.2 Let $n \geq 4$. Then the 3-extra diagnosability of the locally twisted cube LTQ_n under the PMC model is less than or equal to $4n-6$, i.e., $\tilde{t_3}(LTQ_n) \leq 4n-6$.

Proof: Let A be defined in Lemma 10.1.8, and let $F_1 = N_{LTQ_n}(A)$, $F_2 = A \cup N_{LTQ_n}(A)$. By Lemma 10.1.8, $|F_1| = 4n-9$, $|F_2| = |A| + |F_1| = 4n-5$, $|V(LTQ_n[A])| \geq 4$ and $|V(LTQ_n - F_2)| \geq 4$, F_1 is a 3-extra cut of LTQ_n. Therefore, F_1 and F_2 are 3-extra faulty sets of LTQ_n with $|F_1| = 4n-9$ and $|F_2| = 4n-5$. Since $A = F_1 \triangle F_2$ and $N_{LTQ_n}(A) = F_1 \subset F_2$, there is no edge of LTQ_n between $V(LTQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 10.2.1, we can deduce that LTQ_n is not 3-extra $(4n-5)$-diagnosable under PMC model. Hence, by the definition of 3-extra diagnosability, we conclude that the 3-extra diagnosability of LTQ_n is less than $4n-5$, i.e., $\tilde{t_3}(LTQ_n) \leq 4n-6$. □

Lemma 10.2.3 Let $n \geq 5$. Then the 3-extra diagnosability of the locally twisted cube LTQ_n under the PMC model is more than or equal to $4n-6$, i.e., $\tilde{t_3}(LTQ_n) \geq 4n-6$.

Proof: By the definition of 3-extra diagnosability, it is sufficient to show that LTQ_n is 3-extra $(4n-6)$-diagnosable. By Theorem 10.2.1, to prove LTQ_n is 3-extra $(4n-6)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(LTQ_n)$ with $u \in V(LTQ_n) \setminus (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of 3-extra faulty subsets F_1 and F_2 of $V(LTQ_n)$ with $|F_1| \leq 4n-6$ and $|F_2| \leq 4n-6$.

Suppose, by way of contradiction, that there are two distinct 3-extra faulty subsets F_1 and F_2 of LTQ_n with $|F_1| \leq 4n-6$ and $|F_2| \leq 4n-6$, but the vertex set pair (F_1, F_2) is not satisfied with the condition in Theorem 10.2.1, i.e., there are no edges between $V(LTQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$.

Assume $V(LTQ_n) = F_1 \cup F_2$. Since $n \geq 5$, we have that $2^n = |V(LTQ_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq (4n-6) + (4n-6) = 8n-12$, a contradiction. Therefore, $V(LTQ_n) \neq F_1 \cup F_2$.

The following we discuss the case when $F_2 \setminus F_1 \neq \emptyset$ and $V(LTQ_n) \neq F_1 \cup F_2$.

Since there are no edges between $V(LTQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a 3-extra faulty set, $LTQ_n - F_1$ has two parts $LTQ_n - F_1 - F_2$ and $LTQ_n[F_2 \setminus F_1]$. Thus, every component G_i of $LTQ_n - F_1 - F_2$ satisfies $|V(G_i)| \geq 4$ and every component C_i of $LTQ_n[F_2 \setminus F_1]$ satisfies $|V(C_i)| \geq 4$. Similarly, every component C_i' of $LTQ_n[F_1 \setminus F_2]$ satisfies $|V(C_i')| \geq 4$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a 3-extra faulty set. Since there are no edges between $V(LTQ_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is also a 3-extra cut. When $F_1 \setminus F_2 = \emptyset$, $F_1 \cap F_2 = F_1$ is also a 3-extra faulty set. Since there are no edges between $V(LTQ_n - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 3-extra cut. By Theorem 10.1.10, $|F_1 \cap F_2| \geq 4n-9$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 4 + 4n - 9 = 4n-5$, which contradicts with that $|F_2| \leq 4n-6$. So LTQ_n is 3-extra $(4n-6)$-diagnosable. By the definition of $\tilde{t_3}(LTQ_n)$, $\tilde{t_3}(LTQ_n) \geq 4n-6$. The proof is completed. □

Combining Lemmas 10.2.2 and 10.2.3, we have the following theorem.

Theorem 10.2.4 Let $n \geq 5$. Then the 3-extra diagnosability of the locally twisted cubes LTQ_n under the PMC model is $4n-6$.

10.3 The 3-Extra Diagnosability of the Locally Twisted Cube under the MM* Model

Before discussing the 3-extra diagnosability of the locally twisted cube LTQ_n under the MM* model, we firstly give the necessary and sufficient condition of that a system (graph) G is g-extra t-diagnosable under MM* model.

Theorem 10.3.1 [75, 112, 114] A system $G = (V, E)$ is g-extra t-diagnosable under the MM* model if and only if for each distinct pair of g-extra faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ satisfies one of the following conditions.

(1) There are two vertices $u, w \in V \setminus (F_1 \cup F_2)$ and there is a vertex $v \in F_1 \triangle F_2$ such that $uw \in E$ and $vw \in E$.

(2) There are two vertices $u, v \in F_1 \setminus F_2$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$.

(3) There are two vertices $u, v \in F_2 \setminus F_1$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$.

Firstly we give the lower bound of 3-extra diagnosability of the locally twisted cube LTQ_n under the MM* model, where $n \geq 4$.

Lemma 10.3.2 Let $n \geq 4$. Then the 3-extra diagnosability of the locally twisted cube LTQ_n under the MM* model is less than or equal to $4n-6$, i.e., , $\tilde{t_3}(LTQ_n) \leq 4n-6$.

Proof: Let A be defined in Lemma 10.1.8, and let $F_1 = N_{LTQ_n}(A)$, $F_2 = A \cup N_{LTQ_n}(A)$. By Lemma 10.1.8, $|F_1| = 4n-9$, $|F_2| = |A| + |F_1| = 4n-5$, $|V(LTQ_n[A])| \geq 4$ and $|V(LTQ_n - F_2)| \geq 4$, F_1 is a 3-extra cut of LTQ_n. Therefore, F_1 and F_2 are 3-extra faulty sets of LTQ_n with $|F_1| = 4n-9$ and $|F_2| = 4n-5$. Since $A = F_1 \triangle F_2$ and $N_{LTQ_n}(A) = F_1 \subset F_2$, there is no edge of LTQ_n between $V(LTQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 10.3.1, we can deduce that LTQ_n is not 3-extra $(4n-5)$-diagnosable under MM* model. Hence, by the definition of 3-extra diagnosability, we conclude that the 3-extra diagnosability of LTQ_n is less than $4n-5$, i.e., $\tilde{t_3}(LTQ_n) \leq 4n-6$. $\square$

A component of a graph G is odd or even according as it has an odd or even number of vertices. We denote by $o(G)$ the number of odd components of G.

Lemma 10.3.3 [13] A graph $G = (V, E)$ has a perfect matching if and only if $o(G-S) \leq |S|$ for all $S \subseteq V$.

Secondly we prove the upper bound of the 3-extra diagnosability of the locally twisted cube LTQ_n under the MM* model with $n \geq 7$.

Lemma 10.3.4 Let $n \geq 7$. Then the 3-extra diagnosability of the locally twisted cube LTQ_n under the MM* model is more than or equal to $4n-6$, i.e., $\tilde{t_3}(LTQ_n) \geq 4n-6$.

Proof: By the definition of the 3-extra diagnosability, it is sufficient to show that LTQ_n is 3-extra $(4n-6)$-diagnosable.

By Theorem 10.3.1, suppose, by way of contradiction, that there are two distinct 3-extra faulty subsets F_1 and F_2 of LTQ_n with $|F_1| \leq 4n-6$ and $|F_2| \leq 4n-6$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 10.3.1. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similarly to the discussion on $V(LTQ_n) = F_1 \cup F_2$ in Lemma 10.2.3, we can deduce $V(LTQ_n) \neq F_1 \cup F_2$. Therefore, we have the following discussion for $V(LTQ_n) \neq F_1 \cup F_2$.

Claim 1. $LTQ_n - F_1 - F_2$ has no isolated vertex.

Suppose, by way of contradiction, that $LTQ_n - F_1 - F_2$ has at least one isolated vertex w. Since F_1 is a 3-extra faulty set, there are at least one vertex $u \in F_2 \setminus F_1$ such that u are adjacent to w. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 10.3.1, by the condition (3) of Theorem 10.3.1, there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Therefore, there is just a vertex u is adjacent to w.

Case 1 . $F_1 \setminus F_2 = \emptyset$.

If $F_1 \setminus F_2 = \emptyset$, then $F_1 \subseteq F_2$. Since F_2 is a 3-extra faulty set, every component G_i of $LTQ_n - F_1 - F_2$ has $|V(G_i)| \geq 4$. Thus, $LTQ_n - F_1 - F_2$ has no isolated vertex.

Case 2 . $F_1 \setminus F_2 \neq \emptyset$.

Similarly, since $F_1 \setminus F_2 \neq \emptyset$, by the condition (2) of Theorem 10.3.1 and the hypothesis, we can deduce that there is just a vertex $v \in F_1 \setminus F_2$ such that v is adjacent to w.

Let $W \subseteq V(LTQ_n) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $LTQ_n[V(LTQ_n) \setminus (F_1 \cup F_2)]$, and H be the induced subgraph by the vertex set $V(LTQ_n) \setminus (F_1 \cup F_2 \cup W)$. Then for any $w \in W$, there are $(n-2)$ neighbors in $F_1 \cap F_2$. By Lemmas 10.3.3 and 10.1.3, $|W| \leq o(LTQ_n - (F_1 \cup F_2)) \leq |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq (4n-6) + (4n-6) - (n-2) = 7n-10$. Assume $V(H) = \emptyset$. Then $2^n = |V(LTQ_n)| = |F_1 \cup F_2| + |W| = |F_1| + |F_2| - |F_1 \cap F_2| \leq (4n-6) + (4n-6) - (n-2) + (7n-10) = 14n-20$, a contradiction to that $n \geq 7$. So $V(H) \neq \emptyset$.

The following we discuss the case when $F_1 \setminus F_2 \neq \emptyset$, $F_2 \setminus F_1 \neq \emptyset$ and $V(H) \neq \emptyset$.

Since the vertex set pair (F_1,F_2) is not satisfied with the condition (1) of Theorem 10.3.1, and there are not isolated vertices in H, we induce that there is no edge between $V(H)$ and $F_1 \triangle F_2$. Note that $F_2 \backslash F_1 \neq \emptyset$. If $F_1 \cap F_2 = \emptyset$, then this is a contradiction to that LTQ_n is connected. Therefore, $F_1 \cap F_2 \neq \emptyset$. Thus, $F_1 \cap F_2$ is a vertex cut of LTQ_n. Since F_1 is a 3-extra faulty set of LTQ_n, we have that every component H_i of H has $|V(H_i)| \geq 4$ and every component C_i of $LTQ_n[W \cup (F_2 \backslash F_1)]$ has $|V(C_i)| \geq 4$. Since F_2 also is a 3-extra faulty set of LTQ_n, we have that every component C_i' of $LTQ_n[W \cup (F_1 \backslash F_2)]$ has $|V(C_i')| \geq 4$. Note that $LTQ_n - (F_1 \cap F_2)$ has two parts: H and $LTQ_n[W \cup (F_1 \triangle F_2)]$. Let $b_i \in V(LTQ_n[W \cup (F_1 \triangle F_2)])$. If $b_i \in W$, then b_i has two neighbors $u \in V(C_i)$ and $v \in V(C_i')$. Then $b_i \in V(C_i \cup C_i')$ and $|V(C_i \cup C_i')| \geq 4$. Thus, $F_1 \cap F_2$ is a 3-extra cut of LTQ_n. By Theorem 10.1.10, $|F_1 \cap F_2| \geq 4n-9$. Since $|V(C_i)| \geq 4$, $|F_2 \backslash F_1| \geq 3$. Since $|F_1 \cap F_2| = |F_2| - |F_2 \backslash F_1)| \leq (4n-6) - 3 = 4n-9$, we have $|F_1 \cap F_2| = 4n-9$. Then $|F_2 \backslash F_1| = 3$ and $|F_2| = 4n-6$. Similarly, $|F_1 \backslash F_2| = 3$, $|F_1| = 4n-6$. By Theorem 10.1.13, the locally twisted cube LTQ_n is tightly $(4n-9)$ super-3-extra-connected, i.e., $LTQ_n - (F_1 \cap F_2)$ has two components, one of which is a subgraph of order 4. Noted that $|W| \leq 7n-10$. $2^n = |V(LTQ_n)| = |F_1 \backslash F_2| + |F_2 \backslash F_1| + |F_1 \cap F_2| + |V(H)| + |W| \leq 3+3+(4n-9)+4+(7n-10) = 11n-9$, a contradiction to $n \geq 7$. Therefore, $LTQ_n - F_1 - F_2$ has no isolated vertex when $F_1 \setminus F_2 \neq \emptyset$, $F_2 \setminus F_1 \neq \emptyset$ and $V(H) \neq \emptyset$. The proof of Claim 1 is completed.

Let $u \in V(LTQ_n) \setminus (F_1 \cup F_2)$. By Claim 1, u has at least one neighbor vertex in $LTQ_n - F_1 - F_2$. Since the vertex set pair (F_1,F_2) is not satisfied with any one condition in Theorem 10.3.1, by the condition (1) of Theorem 10.3.1, for any pair of adjacent vertices $u,w \in V(LTQ_n) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(LTQ_n)$ and $uv \in E(LTQ_n)$. It follows that u has no neighbor vertex in $F_1 \triangle F_2$. By the arbitrariness of u, there is no edge between $V(LTQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a 3-extra faulty set, $|F_2 \setminus F_1| \geq 4$ and $|V(LTQ_n - F_2 - F_1)| \geq 4$. Since F_1 also is 3-extra faulty sets, $|F_1 \setminus F_2| \leq 4$ and $|V(LTQ_n - F_1 - F_2)| \geq 4$. Then $F_1 \cap F_2$ is a 3-extra cut of LTQ_n. By Theorem 10.1.10, we have $|F_1 \cap F_2| \geq 4n-9$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 4+(4n-9) = 4n-5$, which contradicts $|F_2| \leq 4n-6$. Therefore, LTQ_n is 3-extra $(4n-6)$-diagnosable and $\tilde{t_3}(LTQ_n) \geq 4n-6$. The proof is completed. □

Combining Lemmas 10.3.2 and 10.3.4, we have the following theorem.

Theorem 10.3.5 Let $n \geq 7$. Then the 3-extra diagnosability of the locally twisted cube LTQ_n under the MM* model is $4n-6$.

Chapter 11

Diagnosability of Cayley Graphs Generated by Transposition Trees under the MM* Model

In this chapter, it is proved that diagnosability of $Cay(T_n, S_n)$ is $n-1$ under the MM* model for $n \geq 4$. The results in this chapter is published in Annals of Applied Mathematics [90].

11.1 Definitions & Notations

Given a system $G = (V, E)$ and the comparison scheme $M(V(G), L)$, for a vertex $u \in V$, let X_u be the set of vertices such that $X_u = \{v :$ either $uv \in E$ or $(u, v)_w \in L\}$. That is, a vertex in X_u is either linked to u or compared with u by some other vertex. Let Y_u be the set of edges among vertices of X_u, such that $Y_u = \{vw : v, w \in X_u$ and $(u, v)_w \in L\}$. Let $G_u = (X_u, Y_u)$.

For a vertex $u \in V$, the cardinality of a minimum vertex cover of G_u is called the *order of vertex u*.

Denote $T(X)$ to be the set of vertices that are outside of X and are compared to some vertices of X by some vertices of X (Fig. 11.1). Given G and $M(V(G), L)$, for a subset of vertices $X \subseteq V$,

$$T(X) = \{u : (u, v)_w \in L \text{ and } v, w \in X \text{ and } u \notin X.\}$$

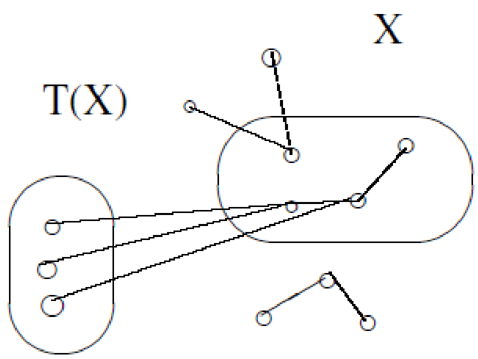

Fig. 11.1 An example of X and $T(X)$

Here we give the definition of components-composition graphs as follow.

Definition 11.1.1 ([18]) The class of m-dimensional components-composition graphs, denoted by CCG_m, is defined recursively as follows: 1) $CCG_1 = \{K_1\}$. 2) Let $m \geq 2$ be a positive integer. Given l $CCG_{m-1}s$ $G_1, G_2, \ldots, G_l$, where

$$\nu(G_i) \leq \sum_{1 \leq j \leq l, j \neq i} \nu(G_j) \text{ and } 2 \leq l \leq \frac{\sum_{i=1}^{l} \nu(G_i)}{2} + 1,$$

a connected graph G constructed from $G_1, G_2, \ldots, G_l$ by adding a perfect matching PM in $\{xy : x \in V(G_i) \text{ and } y \in V(G_j) \text{ for } 1 \leq i, j \leq l \text{ and } i \neq j\}$ is a graph in CCG_m. For convenience, we use the notation $PM(G_1, G_2, \ldots, G_l)$ to represent such a graph. Note that $V(G) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_l)$ and $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_l) \cup PM$.

11.2 Relationship between m-Dimensional Components - Composition Graphs & Cayley Graphs Generated by Transposition Trees

Let T_n be a transposition tree and let $i \in V(T_n)$. Adding a new vertex $n+1$ and an edge $i(n+1)$ to T_n, we obtain a new transposition tree, denoted by T_{n+1}.

Theorem 11.2.1 If $Cay(T_n,S_n) \in CCG_n$, then $Cay(T_{n+1},S_{n+1}) \in CCG_{n+1}$.

Proof: We decompose S_{n+1} by the last position. Let H_i be defined as above. Then H_i and $Cay(T_n,S_n)$ are isomorphic, where $i = 1,2,\ldots,n+1$. It is easy to see that all cross-edges are a perfect matching PM of $Cay(X_{n+1},S_{n+1})$. Therefore, $Cay(X_{n+1},S_{n+1}) = PM(H_1,H_2,\ldots,H_{n+1}) \in CCG_{n+1}$. □

Let $T_n(\geq 3)$ be a transposition tree and let v be a vertex of degree one in T_n. Then $T_n - \{v\}$ is still a transposition tree. Repeating above procedures, we can obtain a transposition tree T_3. Note that $Cay(T_3,S_3) \in CCG_3$. By Theorem 11.2.1, we have the following theorem.

Theorem 11.2.2 $Cay(T_n,S_n) \in CCG_n$.

11.3 Diagnosability of Cayley Graphs Generated by Transposition Trees under the MM* Model

In this section, we will give the diagnosability of Cayley graphs generated by transposition trees under the MM* model.

The following two theorems show the structure of G.

Theorem 11.3.1 [52] Let $t \geq 3$ be a positive integer and let $G_1,G_2,\ldots,G_l$ be l components of a CCG, $G = PM(G_1,G_2,\ldots,G_l)$. Then, G is $(t+1)$-diagnosable under the MM* model if, for each $i \in \{1,2,\ldots,l\}$, the following three conditions hold: (1) $orderG_i(v) \geq t$ for each vertex $v \in V(G_i)$; (2) $\nu(V(G_i)) \geq 2^t$; and (3) $\kappa(G_i) \geq t$.

Theorem 11.3.2 [56] (P. Hall's theorem) Let $G = (U;W)$ be a bipartite graph. Then G has a matching covering U if and only if $|N(X)| \geq |X|$ for all $X \subseteq U$.

Proposition 11.3.1 Let $n \geq 3$ be a positive integer. Then a vertex of $Cay(T_n,S_n)$ has order $n-1$, where for a vertex $u \in V$, the cardinality of a minimum vertex cover of G_u is called the order of vertex u..

Proof: By Lemma 2.4.5, without loss of generality, it is sufficient to check the order for a vertex $u = (1)$. By the definition of $Cay(T_n, S_n)_u = (X_u, Y_u)$, X_u consists of those vertices that are either linked to u, denoted by X_1, or being compared to u, denoted by X_2. So, X_u is the union of two sets X_1 and X_2. The total number of vertices in X_1 is $n-1$, and the total number of vertices in X_2 is at most $(n-1)(n-2)$. Y_u consists of all edges vw such that w compares u and v, i.e., w is linked to u and v is linked to w. That is, $Y_u = \{vw : w \in X_1, v \in X_2\}$. It can be seen that $Cay(T_n, S_n)_u$ is a bipartite graph. To find the order of u, we need to find the size of the minimum vertex cover. From the Konig-Egervary theorem, in a bipartite graph, the size of the minimum vertex cover is equal to the size of the maximum matching. A matching is a set of edges of the graph such that no two edges in the set share a common vertex. The matching is maximum if it has the maximum number of edges over all matchings in the graph.

Claim. Let $v, w \in X_1$ with $v \neq w$. Then $|N(v) \cap N(w)| \leq 2$.

In this case, $u = (1) \in N(v) \cap N(w)$. Suppose, on the contrary, that $|N(v) \cap N(w)| \geq 3$. Let $a, b \in N(v) \cap N(w)$ with $a \neq u$ and $b \neq u$. Then $uvawu$ and $uvbwu$ are a cycle of length 4. Since $u = (1)$, v and w are two transpositions. Let $v = (ij)$ and $w = (rt)$. Since $uvawu$ is a cycle of length 4, $a = (ij)(rt)$, and (ij) and (rt) are disjoint. Thus, $b = (ij)(rt)$. This is a contradiction to $a \neq b$. The proof of this claim is completed.

Since $n \geq 3$, we have $|X_1| \geq 2$. Let $x_1, y_1 \in X_1$. By the Claim, we have $|X_2| \geq |N(\{x_1, y_1\})| \geq 2(n-2) - 1 \geq n-1 = |X_1|$. Let $X \subseteq X_1$. For $2 \leq |X| \leq n-1$, $|N(X)| \geq |N(\{x_1, y_1\})| \geq 2(n-2) - 1 \geq n-1 \geq |X|$. When $|X| = 1$, we have $|N(X)| \geq |X|$. Thus, by Theorem 11.3.2, $Cay(T_n, S_n)_u$ has a maximum matching covering X_1 and the size of the maximum matching for $Cay(T_n, S_n)_u$ is $(n-1)$, which is also the order of u. The proof is completed. □

Now we are ready to show the main results.

Theorem 11.3.3 Cayley graphs $Cay(T_n, S_n)$ generated by transposition trees T_n is $(n-1)$-diagnosable under the MM* model for $n \geq 4$.

Proof: By Theorem 11.2.2, $Cay(T_n,S_n) = PM(\underbrace{Cay(T_{n-1},S_{n-1}),\ldots,Cay(T_{n-1},S_{n-1})}_{n})$. By Proposition 11.3.1, $order_{Cay(T_{n-1},S_{n-1})}(v) \geq n-2$ for each vertex $v \in S_{n-1}$ where $n \geq 4$. By the definition of $Cay(T_{n-1},S_{n-1})$, $|S_{n-1}| = (n-1)! \geq 2^{n-2}$ for $n \geq 4$. By Lemma 2.4.4, $\kappa(Cay(T_n,S_n)) = n-2$. Thus, by Theorem 11.3.1 $Cay(T_n,S_n)$ is $(n-2)+1 = (n-1)$-diagnosable for $n \geq 4$. □

There are several different ways to characterize a t-diagnosable system under the comparison approach [75]. In this study, we use one particular characterization given in [75] which gives the three sufficient conditions for a system to be t-diagnosable.

Finally, we point out that $Cay(T_4,S_4)$ is the least $Cay(T_n,S_n)$ satisfying the three sufficient conditions in Theorem 11.3.1. Because $Cay(T_3,S_3)$ is isomorphic to the star graph, by [116] $Cay(T_3,S_3)$ is not 2-diagnosable.

Theorem 11.3.4 [51] Let $G = (V,E)$ be a graph representation of a system, where V represents the processors and E represents their interconnections. Then, $d(G) \leq \delta(G)$ under the MM* model.

Theorem 11.3.5 Diagnosability of $Cay(T_n,S_n)$ is $n-1$ under the MM* model for $n \geq 4$.

Proof: By Theorem 11.3.3, $d(Cay(T_n,S_n)) \geq n-1$ for $n \geq 4$. Because $Cay(T_n,S_n)$ $(n \geq 1)$ is regular with the common degree $n-1$, $\delta(Cay(T_n,S_n)) = n-1$. By Theorem 11.3.4, $d(Cay(T_n,S_n)) \leq \delta(Cay(T_n,S_n)) = n-1$. Therefore, $d(Cay(T_n,S_n)) = n-1$ for $n \geq 4$. □

11.4 Conclusion

The diagnosability of Cayley graph network $Cay(T_n,S_n)$ generated by transposition trees under the MM* model was studied here. Under this model, the system is self-diagnosable if we know the diagnosability of the system. We proved that a system with the $Cay(T_n,S_n)$ structure is $(n-1)$-diagnosable under the MM* model if $n \geq 4$. Based on the result, a polynomial-time algorithm proposed in [75] can be directly used to diagnose the system if there are at most $(n-1)$ faulty processors. The diagnosis involves only one test phase to identify the faulty processors and one repair or replacement phase. Thus it is applicable in the environment that the components are reliable and periodic and quick testings are affordable. Furthermore, the algorithm can be used as a component of a larger diagnosis scheme to perform a given phase of fault location, as opposed to being used as a stand-alone diagnosis tool.

Chapter 12

The g-Good-Neighbor & g-Extra Diagnosability of Networks

In this chapter, we show the relationship between the g-good-neighbor (extra) diagnosability and g-good-neighbor (extra) connectivity of graphs. The results in this chapter was accepted by Theoretical Computer Science [94].

12.1 The Relationship between the g-Extra Diagnosability & the g-Extra Connectivity under the PMC Model & MM* Model

Firstly we give two existing propositions on the relationship between the g-good-neighbor connectivity and g-extra connectivity.

Proposition 12.1.1 [71] Let G be a g-extra and g-good-neighbor connected graph. Then $\tilde{\kappa}^{(g)}(G) \leq \kappa^{(g)}(G)$.

Proposition 12.1.2 [71] Let G be a nature connected graph. Then $\kappa^{*}(G) = \tilde{\kappa}^{(1)}(G)$.

Before discussing the g-extra diagnosability of networks under the PMC model and MM* model, see the Theorem 5.2.1, 5.3.1, 10.2.1 and 10.3.1, which show the necessary and sufficient conditions of that a system (graph) G is g-extra (g-good-neighbor) t-diagnosable under the PMC and MM* model.

Theorem 12.1.1 Let $G=(V(G),E(G))$ be a g-extra connected graph. If there is connected subgraph H of G with $|V(G)|=g+1$ such that $N(V(H))$ is a minimum g-extra cut of G, then the g-extra diagnosability of G is less than or equal to $\tilde{\kappa}^{(g)}(G)+g$, i.e., $\tilde{t}_g(G)\leq\tilde{\kappa}^{(g)}(G)+g$ under the PMC model and MM* model.

Proof: Since $N(V(H))$ is a minimum g-extra cut of G, $|N(V(H))|=\tilde{\kappa}^{(g)}(G)$ holds. Let $F_1=N(V(H))$, and let $F_2=F_1\cup V(H)$. Then $|F_2|=\tilde{\kappa}^{(g)}(G)+g+1$. Therefore, F_1 and F_2 are both g-extra faulty sets of G with $|F_1|=\tilde{\kappa}^{(g)}(G)$ and $|F_2|=\tilde{\kappa}^{(g)}(G)+g+1$. Since $V(H)=F_1\bigtriangleup F_2$ and $F_1\subset F_2$, there is no edge of G between $V(G)\backslash(F_1\cup F_2)$ and $F_1\bigtriangleup F_2$. By Theorems 10.2.1 and 10.3.1, we can deduce that G is not g-extra $(\tilde{\kappa}^{(g)}(G)+g+1)$-diagnosable under the PMC model and MM* model. Hence, by the definition of g-extra diagnosability, we conclude that the g-extra diagnosability of G is less than to $\tilde{\kappa}^{(g)}(G)+g+1$, i.e., $\tilde{t}_g(G)\leq\tilde{\kappa}^{(g)}(G)+g$. □

Theorem 12.1.2 Let $G=(V(G),E(G))$ be a g-extra connected graph, and let $V(G)\neq F_1\cup F_2$ for each distinct pair of g-extra faulty subsets F_1 and F_2 of G with $|F_1|\leq\tilde{\kappa}^{(g)}(G)+g$ and $|F_2|\leq\tilde{\kappa}^{(g)}(G)+g$. Then the g-extra diagnosability of G is more than or equal to $\tilde{\kappa}^{(g)}(G)+g$, i.e., $t_g(G)\geq\tilde{\kappa}^{(g)}(G)+g$ under the PMC model.

Proof: By the definition of g-extra diagnosability, it is sufficient to show that G is g-extra $(\tilde{\kappa}^{(g)}(G)+g)$-diagnosable. By Theorem 10.2.1, suppose, on the contrary, that there are two distinct g-extra faulty subsets F_1 and F_2 of G with $|F_1|\leq\tilde{\kappa}^{(g)}(G)+g$ and $|F_2|\leq\tilde{\kappa}^{(g)}(G)+g$, but the vertex set pair (F_1,F_2) does not satisfy the condition in Theorem 10.2.1, i.e., there are no edges between $V(G)\backslash(F_1\cup F_2)$ and $F_1\bigtriangleup F_2$. Without loss of generality, assume that $F_2\setminus F_1\neq\emptyset$.

Since there are no edges between $V(G)\setminus(F_1\cup F_2)$ and $F_1\bigtriangleup F_2$, and F_1 is a g-extra faulty set, $G-F_1$ has two parts $G-F_1-F_2$ and $G[F_2\setminus F_1]$ (for convenience). Thus, every component G_i of $G-F_1-F_2$ has $|V(G_i)|\geq g+1$ and every component B_i' of $G[F_2\setminus F_1]$) has $|V(B_i')|\geq g+1$. Similarly, every component B_i'' of $G[F_1\setminus F_2]$) has $|V(B'')|\geq g+1$ when

$F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a g-extra faulty set of G. Note that $F_1 \cap F_2 = F_1$ is also a g-extra faulty set when $F_1 \setminus F_2 = \emptyset$. Since there are no edges between $V(G-F_1-F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a g-extra cut of G. If $F_1 \cap F_2 = \emptyset$, this is a contradiction to that G is connected. Therefore, $F_1 \cap F_2 \neq \emptyset$. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq g+1+\tilde{\kappa}^{(g)}(G)$, which contradicts with that $|F_2| \leq \tilde{\kappa}^{(g)}(G)+g$. So G is g-extra $(\tilde{\kappa}^{(g)}(G)+g)$-diagnosable. By the definition of $\tilde{t}_g(G)$, $\tilde{t}_g(G) \geq \tilde{\kappa}^{(g)}(G)+g$. □

By Theorems 12.1.1 and 12.1.2, we have the following theorem.

Theorem 12.1.3 Let $G=(V(G),E(G))$ be a g-extra connected graph, and let $V(G) \neq F_1 \cup F_2$ for each distinct pair of g-extra faulty subsets F_1 and F_2 of G with $|F_1| \leq \tilde{\kappa}^{(g)}(G)+g$ and $|F_2| \leq \tilde{\kappa}^{(g)}(G)+g$. If there is connected subgraph H of G with $|V(H)| = g+1$ such that $N(V(H))$ is a minimum g-extra cut of G, then the g-extra diagnosability of G is $\tilde{\kappa}^{(g)}(G)+g$ under the PMC model.

The following results have been obtained in [96].

Lemma 12.1.4 [96] Let BS_n be the bubble-sort star graph and $A = \{(1),(12),(123)\}$. If $n \geq 5$, $F_1 = N(A)$, $F_2 = A \cup N(A)$, then $|F_1| = 6n-15$, $|F_2| = 6n-12$, F_1 is a 2-extra cut of BS_n, and $BS_n - F_1$ has two components $BS_n - F_2$ and $BS_n[A]$.

Theorem 12.1.5 [96] For $n \geq 5$, the 2-extra connectivity of the bubble-sort star graph BS_n is $6n-15$.

By Lemma 12.1.4, there is connected subgraph $BS_n[A]$ of order 3 such that $N(A)$ is a minimum 2-extra cut of BS_n. By Theorem 12.1.5, $\tilde{\kappa}^{(2)}(BS_n) = 6n-15$. Since $n! > [(6n-15)+2]+[(6n-15)+2]$ when $n \geq 5$, we have $V(BS_n) \neq F_1 \cup F_2$ for each distinct pair of 2-extra faulty subsets F_1 and F_2 of BS_n with $|F_1| \leq 6n-15+2$ and $|F_2| \leq 6n-15+2$. By Theorem 12.1.3, we have the following corollary.

Corollary 12.1.6 [96] For $n \geq 5$, the 2-extra diagnosability of the bubble-sort star graph BS_n is $6n-13$ under the PMC model.

Let $G = (V(G), E(G))$ be a g-extra connected graph. Let $W \subseteq V(G) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $G[V(G) \setminus (F_1 \cup F_2)]$, and let H be the induced subgraph by the vertex set $V(G) \setminus (F_1 \cup F_2 \cup W)$ for each distinct pair of g-extra faulty subsets F_1 and F_2 of G with $|F_1| \leq \tilde{\kappa}^{(g)}(G) + g - 1$ and $|F_2| \leq \tilde{\kappa}^{(g)}(G) + g - 1$.

Theorem 12.1.7 Let G be a g-extra connected graph, and let $V(H) \neq \emptyset$ for each distinct pair of g-extra faulty subsets F_1 and F_2 of G with $|F_1| \leq \tilde{\kappa}^{(g)}(G)+g-1$ and $|F_2| \leq \tilde{\kappa}^{(g)}(G)+g-1$. Then the g-extra diagnosability of G is more than or equal to $\tilde{\kappa}^{(g)}(G) + g - 1$, i.e., $t_g(G) \geq \tilde{\kappa}^{(g)}(G) + g - 1$ under the MM* model.

Proof: By the definition of g-extra diagnosability, it is sufficient to show that G is g-extra $(\tilde{\kappa}^{(g)}(G) + g - 1)$-diagnosable.

Suppose, on the contrary, that there are two distinct g-extra faulty subsets F_1 and F_2 of G with $|F_1| \leq \tilde{\kappa}^{(g)}(G) + g - 1$ and $|F_2| \leq \tilde{\kappa}^{(g)}(G) + g - 1$, but the vertex set pair (F_1, F_2) does not satisfy any condition in Theorem 10.3.1. Without loss of generality, suppose that $F_2 \setminus F_1 \neq \emptyset$. Let $W \subseteq V(G) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $G[V(G) \setminus (F_1 \cup F_2)]$, and let H be the induced subgraph by the vertex set $V(G) \setminus (F_1 \cup F_2 \cup W)$. Then $V(H) \neq \emptyset$. We consider the following cases.

Case 1. $g = 0$.

Note that $V(H) \neq \emptyset$ for each distinct pair of 0-good-neighbor faulty subsets F_1 and F_2 of G with $|F_1| \leq \tilde{\kappa}^{(0)}(G) - 1$ and $|F_2| \leq \tilde{\kappa}^{(0)}(G) - 1$ and $F_2 \setminus F_1 \neq \emptyset$. Since the vertex set pair (F_1, F_2) does not satisfy the condition (1) of Theorem 10.3.1, and any vertex of $V(H)$ is not isolated in H, we deduce that there is no edge between $V(H)$ and $F_1 \triangle F_2$. Therefore, $F_1 \cap F_2$ is a 0-good-neighbor cut of G. Thus, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + \tilde{\kappa}^{(0)}(G)$, which contradicts $|F_2| \leq \tilde{\kappa}^{(0)}(G) - 1$.

Case 2. $g \geq 1$.

Claim 1. $G - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $G - F_1 - F_2$ has at least one isolated vertex w_1. Since F_1 is a g-extra faulty set, there is a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w_1. Meanwhile, since the vertex set pair (F_1, F_2) does not satisfy any condition in Theorem 10.3.1, by the condition (3) of Theorem 10.3.1, there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w_1. Thus, there is just a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w_1. If $F_1 \setminus F_2 = \emptyset$, then $F_1 \subseteq F_2$. Since F_2 is a g-extra faulty set, every component G_i of $G - F_1 - F_2 = G - F_2$ satisfies $|V(G_i)| \geq g+1$. Therefore, $G - F_1 - F_2$ has no isolated vertex for $g \geq 1$. Thus, $F_1 \setminus F_2 \neq \emptyset$. Similarly, we know that there is just a vertex $a \in F_1 \setminus F_2$ such that a is adjacent to w_1. Let $W \subseteq V(G) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $G[V(G) \setminus (F_1 \cup F_2)]$, and let H be the induced subgraph by the vertex set $V(G) \setminus (F_1 \cup F_2 \cup W)$. Then $V(H) \neq \emptyset$.

Since the vertex set pair (F_1, F_2) does not satisfy the condition (1) of Theorem 10.3.1, and none of the vertices in V(H) is isolated vertex in H, we know that there is no edge between $V(H)$ and $F_1 \triangle F_2$. Note $F_2 \backslash F_1 \neq \emptyset$. If $F_1 \cap F_2 = \emptyset$, then this is a contradiction to that G is connected. Therefore, $F_1 \cap F_2 \neq \emptyset$. Thus, $F_1 \cap F_2$ is a vertex cut of G. Since F_1 is a g-extra faulty set of G, we have that every component H_i of H satisfies $|V(H_i)| \geq g+1$ and every component B_i of $G[W \cup (F_2 \setminus F_1)])$ satisfies $|V(B_i)| \geq g+1$. Since F_2 is a g-extra faulty set of G, we have that every component B_i' of $G[W \cup (F_1 \setminus F_2)])$ has $|V(B_i')| \geq g+1$. Note that $G - (F_1 \cap F_2)$ has two parts (for convenience): H and $G[W \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_1)])$. Let $\mathscr{B}_i$ be a component of $G[W \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_1)])$ and let $b_i \in V(\mathscr{B}_i)$. If $b_i \in W$, then there is a component G_i of $G([F_2 \setminus F_1])$ ($|V(G_i)| \geq g+1$) and a component B_i of $G([F_1 \setminus F_2])$ ($|V(B_i)| \geq g+1$) such that $b_i \in V(G_i)$ and $b_i \in V(B_i)$. It follows that $G_i \cup B_i$ is connected in $G[W \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_1)])$ and $b_i \in V(G_i \cup B_i)$. Since a connection is an equivalence relation on the vertex set $W \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$, $\mathscr{B}_i = (G_i \cup B_i)$ holds. Therefore, $|V(\mathscr{B}_i)| \geq g+1$. If $b_i \in (F_2 \setminus F_1)$, then there is a component G_i of $G([F_2 \setminus F_1])$ ($|V(G_i)| \geq g+1$) such that $b_i \in V(G_i)$. It follows that G_i is connected in $G[W \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_1)])$ and $b_i \in V(G_i)$. Since a connection is an equivalence relation on the vertex set $W \cup (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$, we have that G_i is a subgraph of $\mathscr{B}_i$. Therefore, $|V(\mathscr{B}_i)| \geq g+1$. Similarly, if $b_i \in (F_1 \setminus F_2)$, then $|V(\mathscr{B}_i)| \geq g+1$. Therefore, $F_1 \cap F_2$ is a g-extra cut of G.

Since every component B_i of $G[W \cup (F_2 \setminus F_1)])$ has $|V(B_i)| \geq g+1$, we have $|F_2 \setminus F_1| \geq g$ and we have that $\tilde{\kappa}^{(g)}(G) + g - 1 \geq |F_2| = |F_1 \cap F_2| + |F_2 \setminus F_1| \geq \tilde{\kappa}^{(g)}(G) + g$, a contradiction. The proof of Claim 1 is completed.

Let $u \in V(G) \setminus (F_1 \cup F_2)$. By Claim 1, u has at least one neighbor vertex in $G - F_1 - F_2$. Since the vertex set pair (F_1, F_2) does not satisfy any one condition in Theorem 10.3.1, by the condition (1) of Theorem 10.3.1, for any pair of adjacent vertices $u, w \in V(G) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(G)$ and $vw \in E(G)$. It follows that u has no neighbor in $F_1 \triangle F_2$. Since u is taken arbitrarily, so there is no edge between $V(G) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. If $F_1 \cap F_2 = \emptyset$, then this is a contradiction to the assumption that G is connected. Therefore, $F_1 \cap F_2 \neq \emptyset$ and $F_1 \cap F_2$ is a cut of G. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a g-extra faulty set, we have that every component H_i of $G - F_1 - F_2$ has $|V(H_i)| \geq g+1$ and every component G_i of $G([F_2 \setminus F_1])$ has $|V(G_i)| \geq g+1$. Suppose that $F_1 \setminus F_2 = \emptyset$. Then $F_1 \cap F_2 = F_1$. Since F_1 is a g-extra faulty set of G, we have that $F_1 \cap F_2 = F_1$ is a g-extra faulty set of G. Since there is no edge between $V(G) \setminus (F_1 \cup F_2)$ and $F_2 \setminus F_1$, we have that $F_1 \cap F_2 = F_1$ is a g-extra cut of G. Suppose that $F_1 \setminus F_2 \neq \emptyset$. Similarly, every component B_i of $G([F_1 \setminus F_2])$ has $|V(B_i)| \geq g+1$. Note that $G - (F_1 \cap F_2)$ has three parts (for convenience): H, $G[F_1 \setminus F_2]$ and $G[F_2 \setminus F_1]$. Therefore, $F_1 \cap F_2$ is a g-extra cut of G. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq g + 1 + \tilde{\kappa}^{(g)}(G)$, which contradicts $|F_2| \leq \tilde{\kappa}^{(g)}(G) + g - 1$. Therefore, G is g-extra $(\tilde{\kappa}^{(g)}(G) + g - 1)$-diagnosable and $\tilde{t}_g(G) \geq \tilde{\kappa}^{(g)}(G) + g - 1$. The proof is completed. □

By Theorems 12.1.1 and 12.1.7, we have the following theorem.

Theorem 12.1.8 Let G be a g-extra connected graph, and let $V(H) \neq \emptyset$ for each distinct pair of g-extra faulty subsets F_1 and F_2 of G with $|F_1| \leq \tilde{\kappa}^{(g)}(G) + g$ and $|F_2| \leq \tilde{\kappa}^{(g)}(G) + g$. If there is connected subgraph H of G with $|V(H)| = g+1$ such that $N(V(H))$ is a minimum g-extra cut of G, then the g-extra diagnosability of G is $\tilde{\kappa}^{(g)}(G) + g - 1$ or $\tilde{\kappa}^{(g)}(G) + g$ under the MM* model.

12.2 The Relationship between the g-Good-Neighbor Diagnosability & the g-Good-Neighbor Connectivity under the PMC Model & MM* Model

In this section, we will show the relationship between the g-good-neighbor diagnosability and g-good-neighbor connectivity of networks under the PMC and MM* model.

Theorem 12.2.1 Let $G = (V(G), E(G))$ be a g-good-neighbor connected graph, and let H be connected subgraph of G with $\delta(H) = g$ such that it contains $V(G)$ as least as possible, and $N(V(H))$ is a minimum g-good-neighbor cut of G. Then the g-good-neighbor diagnosability of G is less than or equal to $\kappa^{(g)}(G) + |V(H)| - 1$, i.e., $t_g(G) \leq \kappa^{(g)}(G) + |V(H)| - 1$ under the PMC model and MM* model.

Proof: Since $N(V(H))$ is a minimum g-good-neighbor cut of G, $|N(V(H))| = \kappa^{(g)}(G)$ holds. Let $F_1 = N(V(H))$, and let $F_2 = F_1 \cup V(H)$. Then $|F_2| = \kappa^{(g)}(G) + |V(H)|$. Therefore, F_1 and F_2 are both g-good-neighbor faulty sets of G with $|F_1| = \kappa^{(g)}(G)$ and $|F_2| = \kappa^{(g)}(G) + |V(H)|$. Since $V(H) = F_1 \triangle F_2$ and $F_1 \subset F_2$, there is no edge of G between $V(G) \backslash (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorems 10.2.1 and 10.3.1, we know that G is not g-good-neighbor $(\kappa^{(g)}(G) + |V(H)|)$-diagnosable under the PMC model and MM* model. Hence, by the definition of g-good-neighbor diagnosability, we conclude that the g-good-neighbor diagnosability of G is less than to $\kappa^{(g)}(G) + |V(H)|$, i.e., $t_g(G) \leq \kappa^{(g)}(G) + |V(H)| - 1$. □

Theorem 12.2.2 Let $G = (V(G), E(G))$ be a g-good-neighbor connected graph, and let H' be connected subgraph of G with $\delta(H') = g$ such that it contains $V(G)$ as least as possible, and $V(G) \neq F_1 \cup F_2$ for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of G with $|F_1| \leq \kappa^{(g)}(G) + |V(H')| - 1$ and $|F_2| \leq \kappa^{(g)}(G) + |V(H')| - 1$. Then the g-good-neighbor diagnosability of G is more than or equal to $\kappa^{(g)}(G) + |V(H')| - 1$, i.e., $t_g(G) \geq \kappa^{(g)}(G) + |V(H')| - 1$ under the PMC model.

Proof: By the definition of g-good-neighbor diagnosability, it is sufficient to show that G is g-good-neighbor $(\kappa^{(g)}(G) + |V(H')| - 1)$-diagnosable. By Theorem 5.2.1, suppose, on the contrary, that there are two distinct g-good-neighbor faulty subsets F_1 and F_2 of G with $|F_1| \le \kappa^{(g)}(G)+|V(H')|-1$ and $|F_2| \le \kappa^{(g)}(G)+|V(H')|-1$, but the vertex set pair (F_1,F_2) does not satisfy the condition in Theorem 5.2.1, i.e., there are no edges between $V(G)\backslash(F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$.

Since there are no edges between $V(G)\setminus(F_1\cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a g-good-neighbor faulty set, $G-F_1$ has two parts $G-F_1-F_2$ and $G[F_2\setminus F_1]$ (for convenience). Thus, $|N(v)\cap(V\setminus(F_1\cup F_2))| \ge g$ for every vertex v in $V\setminus(F_1\cup F_2)$. By the definition of H', $|V(G-F_1-F_2)| \ge |V(H')|$ holds. Similarly, $|N(v)\cap(F_2\setminus F_1)| \ge g$ for every vertex v in $F_2\setminus F_1$ and $|N(v)\cap(F_1\setminus F_2)| \ge g$ for every vertex v in $F_1\setminus F_2$ when $F_1\setminus F_2 \neq \emptyset$, and $|F_2\setminus F_1| \ge |V(H')|$ and $|F_1\setminus F_2| \ge |V(H')|$ when $F_1\setminus F_2\neq\emptyset$. Therefore, $F_1\cap F_2$ is also a g-good-neighbor faulty set of G. Note that $F_1\cap F_2 = F_1$ is also a g-extra faulty set when $F_1\setminus F_2=\emptyset$. Since there are no edges between $V(G-F_1-F_2)$ and $F_1\triangle F_2$, $F_1\cap F_2$ is a g-good-neighbor cut of G. If $F_1\cap F_2=\emptyset$, this is a contradiction to that G is connected. Therefore, $F_1\cap F_2\neq\emptyset$ and hence $|F_2| = |F_2\backslash F_1|+|F_1\cap F_2| \ge |V(H')|+\kappa^{(g)}(G)$, which contradicts $|F_2| \le \kappa^{(g)}(G)+|V(H')|-1$. So G is g-good-neighbor $(\kappa^{(g)}(G)+|V(H')|-1)$-diagnosable. By the definition of $t_g(G)$, $t_g(G) \ge \kappa^{(g)}(G)+|V(H')|-1$. □

By Theorems 12.2.1 and 12.2.2, we have the following theorem.

Theorem 12.2.3 Let $G=(V(G),E(G))$ be a g-good-neighbor connected graph, and let H be connected subgraph of G $\delta(G)=g$ such that it contains $V(G)$ as least as possible and $N(V(H))$ is a minimum g-good-neighbor cut of G, and let H' be connected subgraph of G with $\delta(G)=g$ such that it contains $V(G)$ as least as possible. If $V(G)\neq F_1\cup F_2$ for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of G with $|F_1| \le \kappa^{(g)}(G)+|V(H')|-1$ and $|F_2| \le \kappa^{(g)}(G)+|V(H')|-1$, then $\kappa^{(g)}(G)+|V(H')|-1 \le t_g(G) \le \kappa^{(g)}(G)+|V(H)|-1$ under the PMC model.

The following two results have been obtained in [97].

Lemma 12.2.4 [97] Let BS_n be the bubble-sort star graph and $A=\{(1),(12),(123),(13)\}$. If $n\ge 5$, $F_1=N(A)$, $F_2=A\cup N(A)$, then $|F_1|=8n-22$, $|F_2|=8n-18$, $\delta(BS_n-F_1)\ge 2$, and $\delta(BS_n-F_2)\ge 2$.

Theorem 12.2.5 [97] For $n \geq 5$, the 2-good-neighbor connectivity of the bubble-sort star graph BS_n is $8n-22$.

By Lemma 12.2.4, there is connected subgraph $BS_n[A]$ of minimum degree 2 such that it contains $V(BS_n)$ as least as possible and $N(A)$ is a minimum 2-good-neighbor cut of BS_n By Theorem 12.2.5, $\kappa^{(2)}(BS_n) = 8n-22$. Since $n! > [(8n-22)+4-1]+[(8n-22)+4-1]$ when $n \geq 5$, we have $V(BS_n) \neq F_1 \cup F_2$ for each distinct pair of 2-good-neighbor faulty subsets F_1 and F_2 of BS_n with $|F_1| \leq (8n-22)+4-1$ and $|F_2| \leq (8n-22)+4-1$. By Theorem 12.2.3, we have the following corollary.

Corollary 12.2.6 [97] For $n \geq 5$, the 2-good-neighbor diagnosability of the bubble-sort star graph BS_n is $8n-19$ under the PMC model.

Let $G = (V(G), E(G))$ be a g-good-neighbor connected graph. Suppose that H' is connected subgraph of G with $\delta(H') = g$ such that it contains $V(G)$ as least as possible. Let $W \subseteq V(G) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $G[V(G) \setminus (F_1 \cup F_2)]$, and let H^* be the induced subgraph by the vertex set $V(G) \setminus (F_1 \cup F_2 \cup W)$ for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of G with $|F_1| \leq \kappa^{(g)}(G) + |V(H')| - 2$ and $|F_2| \leq \kappa^{(g)}(G) + |V(H')| - 2$.

Theorem 12.2.7 Let G be a g-good-neighbor connected graph, and let $V(H^*) \neq \emptyset$ for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of G with $|F_1| \leq \kappa^{(g)}(G) + |V(H')| - 2$ and $|F_2| \leq \kappa^{(g)}(G) + |V(H')| - 2$. Then the g-good-neighbor diagnosability of G is more than or equal to $\kappa^{(g)}(G) + |V(H')| - 2$, i.e., $t_g(G) \geq \kappa^{(g)}(G) + |V(H')| - 2$ under the MM* model.

Proof: By the definition of g-good-neighbor diagnosability, it is sufficient to show that G is g-good-neighbor $(\kappa^{(g)}(G) + |V(H')| - 2)$-diagnosable.

Suppose, on the contrary, that there are two distinct g-good-neighbor faulty subsets F_1 and F_2 of G with $|F_1| \leq \kappa^{(g)}(G) + |V(H')| - 2$ and $|F_2| \leq \kappa^{(g)}(G) + |V(H')| - 2$, but the vertex set pair (F_1, F_2) does not satisfy any condition in Theorem 5.3.1. Without loss of generality, suppose that $F_2 \setminus F_1 \neq \emptyset$. Let $W \subseteq V(G) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in

$G[V(G)\setminus(F_1\cup F_2)]$, and let H^* be the induced subgraph by the vertex set $V(G)\setminus(F_1\cup F_2\cup W)$. Then $V(H^*)\neq\emptyset$. We consider the following cases.

Case 1. $g=0$.

By the definition of H', $|V(H')|=1$. Note that $V(H^*)\neq\emptyset$ for each distinct pair of 0-good-neighbor faulty subsets F_1 and F_2 of G with $|F_1|\le\kappa^{(0)}(G)+|V(H')|-2=\kappa^{(0)}(G)-1$ and $|F_2|\le\kappa^{(0)}(G)+|V(H')|-2=\kappa^{(0)}(G)-1$ and $F_2\setminus F_1\neq\emptyset$. Since the vertex set pair (F_1,F_2) does not satisfy the condition (1) of Theorem 5.3.1, and any vertex of $V(H^*)$ is not isolated in H^*, we deduce that there is no edge between $V(H^*)$ and $F_1\triangle F_2$. Therefore, $F_1\cap F_2$ is a 0-good-neighbor cut of G. Thus, $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\ge 1+\kappa^{(0)}(G)$, which contradicts $|F_2|\le\kappa^{(0)}(G)-1$.

Case 2. $g\ge1$.

Claim 1. $G-F_1-F_2$ has no isolated vertex.

Suppose, on the contrary, that $G-F_1-F_2$ has at least one isolated vertex w_1. Since F_1 is one g-good-neighbor faulty set, there is a vertex $u\in F_2\setminus F_1$ such that u is adjacent to w_1. Meanwhile, since the vertex set pair (F_1,F_2) does not satisfy any one condition in Theorem 5.3.1, by the condition (3) of Theorem 5.3.1, there is at most one vertex $u\in F_2\setminus F_1$ such that u is adjacent to w_1. Thus, there is just a vertex $u\in F_2\setminus F_1$ such that u is adjacent to w_1. So $d(w_1)=1$ in $G[\{w_1\}\cup(F_2\setminus F_1)]$. Since F_1 is a g-good-neighbor faulty set, this is a contradiction when $g\ge2$. Then F_1 is a nature faulty set. If $F_1\setminus F_2=\emptyset$, then $F_1\subseteq F_2$. Since F_2 is a g-good-neighbor faulty set, every vertex v of $G-F_1-F_2=G-F_2$ has $d(v)\ge g$ in $G-F_2$. Therefore, $G-F_1-F_2$ has no isolated vertex for $g\ge1$. Thus, $F_1\setminus F_2\neq\emptyset$. Similarly, we know that there is just a vertex $a\in F_1\setminus F_2$ such that a is adjacent to w_1 and F_2 is a nature faulty set. Let $W\subseteq V(G)\setminus(F_1\cup F_2)$ be the set of isolated vertices in $G[V(G)\setminus(F_1\cup F_2)]$, and let H^* be the induced subgraph by the vertex set $V(G)\setminus(F_1\cup F_2\cup W)$. Then $V(H^*)\neq\emptyset$.

Since the vertex set pair (F_1,F_2) does not satisfy the condition (1) of Theorem 5.3.1, and any vertex of $V(H^*)$ is not isolated in H^*, we know that there is no edge between $V(H^*)$ and $F_1\triangle F_2$. Note $F_2\setminus F_1\neq\emptyset$. If $F_1\cap F_2=\emptyset$, then this is a contradiction to that G is connected. Therefore, $F_1\cap F_2\neq\emptyset$. Thus, $F_1\cap F_2$ is a vertex cut of G. Since F_1 is a nature faulty set of G, we have that every vertex v of H^* has $d_{H^*}(v)\ge1$ and every vertex a of

$G[W\cup(F_2\setminus F_1)])$ has $d(a)\geq 1$ in $G[W\cup(F_2\setminus F_1)])$. Since F_2 is a nature faulty set of G, we have that every vertex b of $G[W\cup(F_1\setminus F_2)])$ has $d(b)\geq 1$ in $G[W\cup(F_1\setminus F_2)])$. Therefore, every vertex x of $G[W\cup(F_1\setminus F_2)\cup(F_2\setminus F_1)])$ has $d(x)\geq 1$ in $G[W\cup(F_1\setminus F_2)\cup(F_2\setminus F_1)])$. Note that $G-(F_1\cap F_2)$ has two parts (for convenience): H^* and $G[W\cup(F_1\setminus F_2)\cup(F_2\setminus F_1)])$. Therefore, $F_1\cap F_2$ is a nature cut of G and hence we have that $\kappa^*(G)+2-2\geq|F_2|=|F_1\cap F_2|+|F_2\setminus F_1|\geq\kappa^*(G)+1$, a contradiction. The proof of Claim 1 is completed.

Let $u\in V(G)\setminus(F_1\cup F_2)$. By Claim 1, u has at least one neighbor vertex in $G-F_1-F_2$. Since the vertex set pair (F_1,F_2) does not satisfy any condition in Theorem 5.3.1, by the condition (1) of Theorem 5.3.1, for any pair of adjacent vertices $u,w\in V(G)\setminus(F_1\cup F_2)$, there is no vertex $v\in F_1\bigtriangleup F_2$ such that $uw\in E(G)$ and $vw\in E(G)$. It follows that u has no neighbor in $F_1\bigtriangleup F_2$. Since u is taken arbitrarily, so there is no edge between $V(G)\setminus(F_1\cup F_2)$ and $F_1\bigtriangleup F_2$. If $F_1\cap F_2=\emptyset$, then this is a contradiction to that G is connected. Therefore, $F_1\cap F_2\neq\emptyset$ and $F_1\cap F_2$ is a cut of G. Since $F_2\setminus F_1\neq\emptyset$ and F_1 is a g-good-neighbor faulty set, we have that every vertex v of $G-F_1-F_2$ has $d(v)\geq g\geq 1$ in $G-F_1-F_2$ and every vertex a of $G([F_2\setminus F_1])$ has $d(a)\geq g\geq 1$ in $G([F_2\setminus F_1])$. By the definition of H', $|F_2\setminus F_1|\geq|V(H')|$. Suppose that $F_1\setminus F_2=\emptyset$. Then $F_1\cap F_2=F_1$. Since F_1 is a g-good-neighbor faulty set of G, we have that $F_1\cap F_2=F_1$ is a g-good-neighbor faulty set of G. Since there is no edge between $V(G)\setminus(F_1\cup F_2)$ and $F_2\setminus F_1$, we have that $F_1\cap F_2=F_1$ is a g-good-neighbor cut of G. Suppose that $F_1\setminus F_2\neq\emptyset$. Similarly, every vertex b of $G([F_1\setminus F_2])$ has $d(b)\geq g$. Note that $G-(F_1\cap F_2)$ has three parts (for convenience): H^*, $G[F_1\setminus F_2]$ and $G[F_2\setminus F_1]$. Therefore, $F_1\cap F_2$ is a g-good-neighbor cut of G and hence $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\geq|V(H')|+\kappa^{(g)}(G)$, which contradicts $|F_2|\leq\kappa^{(g)}(G)+|V(H')|-2$. Therefore, G is g-good-neighbor $(\kappa^{(g)}(G)+|V(H')|-2)$-diagnosable and $t_g(G)\geq\kappa^{(g)}(G)+|V(H')|-2$. The proof is completed. □

By Theorems 12.2.1 and 12.2.7, we have the following theorem.

Theorem 12.2.8 Let G be a g-good-neighbor connected graph, and let H be connected subgraph of G with $\delta(H)=g$ such that it contains $V(G)$ as least as possible, and $N(V(H))$ is a minimum g-good-neighbor cut of G, and let H' be connected subgraph of G with $\delta(G)=g$ such that it contains $V(G)$ as least as possible. If $V(H^*)\neq\emptyset$ for each distinct pair of g-good-neighbor faulty subsets F_1 and F_2 of G with $|F_1| \le \kappa^{(g)}(G) + |V(H')| - 2$ and $|F_2| \le \kappa^{(g)}(G) + |V(H')| - 2$, then $\kappa^{(g)}(G) + |V(H')| - 2 \le t_g(G) \le \kappa^{(g)}(G) + |V(H)| - 1$ under the MM* model.

12.3 Conclusion

Conditional connectivity and conditional diagnosability are two important metrics for fault tolerance of a multiprocessor system. In this chapter, we showed the relationship between the g-good-neighbor (extra) diagnosability and g-good-neighbor (extra) connectivity of networks. It provided a simple way to study the g-good-neighbor (extra) diagnosability of some well-known networks based on the g-good-neighbor (extra) connectivity. Furthermore, clarifying the relationship between these two metrics could help us determine other conditional diagnosability of networks.

Chapter 13

Conclusion

13.1 Contributions of the Thesis

In Chapter 4, we showed that if G is a $\lambda^{(4)}$-connected graph with $\lambda^{(4)}(G) \leq \xi_4(G)$ and the girth $g(G) \geq 8$, and there are not six vertices u_1, u_2, u_3, v_1, v_2 and v_3 in G such that the distance $d(u_i, v_j) \geq 3 \ (1 \leq i, j \leq 3)$, then G is maximally 4-restricted edge-connected.

In Chapter 5, we proved that the nature diagnosability of $C\Gamma_n$ under the PMC model and MM* model is $2n-3$ except that, the bubble-sort graph B_4, where $n \geq 4$, and the nature diagnosability of B_4 under the MM* model is 4.

In Chapter 6, we showed that the 2-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model and MM* model is $g(n-2)-1$, where $n \geq 4$ and g is the girth of $C\Gamma_n$.

In Chapter 7, we showed that the connectivity of CK_n is $\frac{n(n-1)}{2}$, the nature neighbor connectivity of CK_n is n^2-n-2 and the nature diagnosability of CK_n under the PMC model is n^2-n-1 for $n \geq 4$ and under the MM* model is n^2-n-1 for $n \geq 5$.

In Chapter 8, we proved that the nature diagnosability of BS_n is $4n-7$ under the PMC model for $n \geq 4$, the nature diagnosability of BS_n is $4n-7$ under the MM* model for $n \geq 5$.

In Chapter 9, we proved that (1) the connectivity of XQ_n^k is $4n$; (2) the nature connectivity of XQ_n^k is $8n-4$; (3) the nature diagnosability of XQ_n^k under the PMC model and MM* model is $8n-3$ for $n \geq 2$.

In Chapter 10, we showed that LTQ_n is tightly $(4n-9)$ super 3-extra connected for $n \geq 6$ and the 3-extra diagnosability of LTQ_n under the PMC model and MM* model is $4n-6$ for $n \geq 5$ and $n \geq 7$, respectively.

In Chapter 11, we proved that diagnosability of $Cay(T_n, S_n)$ is $n-1$ under the comparison diagnosis model for $n \geq 4$.

In Chapter 12, we showed the relationship between the g-good-neighbor (extra) diagnosability and g-good-neighbor (extra) connectivity of graphs.

In the thesis, I used some recently proposed, practical oriented measurements such as g-good connectivity, g-extra connectivity, g-good-neighbour diagnosability and g-extra diagnosability. These new parameters better measure the robustness of networks, which is also considered as reliability of networks in this thesis.

On the other hand, combining with using the networks of the following advantageous topological properties, Cayley graph is highly symmetric, has well defined hierarchical structure, highly connected and with great fault tolerance.

We had the corresponding results both on the characterization of network structure and the measurement of reliability of networks as we showed above.

13.2 Future Work

13.2.1 g-Good-Neighbor Connectivity & g-Extra Connectivity

We have been working on the g-good-neighbor and g-extra connectivities and diagnosabilities on several poplar structures (graphs) with g is one or two. It is natural to look at larger value for g, to see if we could have obtain results using the similar methods we have employed. We intend to generalize our methods to handle larger value for g. Furthermore, our research so far are focused on Cayley graphs or related graphs due to their well described structure and nice properties such as highly symmetric and well-structured cut sets. We are interested in looking at other graphs, for example Kautz and De Bruijn graph. More specifically, we are working on the following problems:

- The 2-good-neighbor (3-good-neighbor) connectivity & diagnosability of Bubble-sort star graphs;
- The 2-extra-neighbor(3-extra-neighbor) connectivity and & diagnosability of Bubble-sort star graphs;
- The 2-good-neighbor (3-good-neighbor) connectivity & diagnosability of Cayley graphs generated by complete graphs;
- The 2-extra-neighbor(3-extra-neighbor) connectivity & diagnosability of Cayley graphs generated by complete graphs;
- The 2-good-neighbor (3-good-neighbor) connectivity & diagnosability of expanded k-ary n-cubes;
- The 2-extra-neighbor(3-extra-neighbor) connectivity & diagnosability of expanded k-ary n-cubes;
- The g-good-neighbor (g-good-neighbor) connectivity and diagnosability of more generalized k-ary n-cubes;
- The g-extra-neighbor(g-extra-neighbor) connectivity and diagnosability of more generalized k-ary n-cubes;

and we will work on the following problems (related to general graphs):

- Sufficient conditions for graphs to be maximally n-restricted edge-Connected, where $n \geq 5$;
- Sufficient conditions for graphs to be tightly n super-g-extra-connected, where $n \geq 5$.

13.2.2 Measurements of Connectedness in Graphs

In the Chapter 2, we have listed several types of connectivities or connectivity-related parameters, such as restricted connectivity, super connectivity etc. The aim of introducing all these parameters are for better measuring of the reliability of the networks. One of the key consideration is to get around of the so called trivial case, i.e. the cut set isolates a single vertex. Throughout the introduction of all these parameters, the attempt is certainly clear. It is natural to ask if we could introduce a better measurement along the same direction. For example, we could consider the density of the graph, finding dense component means that we have found the weak link, i.e. the connectivity of the graph. In this case, we will not be worried about the trivial cases.

On the other hand, since the connnectedness of graphs is merely one parameter to characterize the fault-tolerance of the network. It is also an interesting problem to find other parameters to characterize how much a graph is fault-tolerant. This might come from the real world applications.

Furthermore, so far, in this thesis, we have only studied the static graphs, i.e. the graph with given structure which will not change over the time. However, in the real world application, most cases we see dynamic networks, i.e. the graph whose structure changes over the time. There are not many studied on the connectedness of such dynamic network. It is our intention to extending our work into dynamic networks.

13.2.3 Other Works

During my PhD study, there are three papers which are focused on the existence of perfect matching and factorization of regular graphs.

- "The maximum forcing number of a polyomino" is published in The Australasian Journal of Combinatorics;
- "Existence of regular factor in dense graph" with cooperation of Prof. Yuqing Lin and Prof. Hongliang Lu is submitted;
- "The factorization of regular graph" is in preparation.

However, since this thesis is mostly focused on the connectivities of graphs, thus I have decided not to include these results in this thesis.

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